Cash flows, discount, asset pricing, portfolios
a brief review

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Outline

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2. Understanding risk and return
3. Three key principles
   - Higher risk must be compensated by higher expected return
   - The volatility of returns needs not be compensated with higher expected returns
   - Under risk neutrality, the price of any asset depends exclusively on the expected value of its cash flows
4. Discounting real and nominal cash flows
5. Pricing domestic and foreign currency denominated assets
6. Law of one price: cross-border example
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For simplicity, we will focus on a two-period (t, t+1) example.
1. Cash flow, discount, asset pricing

An asset \( n \) is a claim to a cash flow \( x_n \). The **price** of an asset can generically be written as the **expected discounted value of its cash flow**:

\[
q_{n,t} = E_t (D_{t,t+1} \cdot x_{n,t+1})
\]

where \( D_{t,t+1} \) is the **discount factor**, and \( E_t \) is the Expectations operator.

Example: Let the cash flow from an equity \( x_n \) take three possible values \([2 \ 1 \ 1]'\) with probabilities \( \psi \) equal to \([.25 \ .25 \ .5]'\). Investors may have different valuation of resources **when** each of these possible cash flows materializes. Consider an investor who discount the cash flow as follows \([.5 \ 1 \ 1]\), i.e. she will have a different discount \( D \) in each of the “three states of the world” when the asset pays \([2,1,1]\). This investor will value the asset:

\[
q_{n,t} = \sum_{i=1}^{3} \psi (i) D_{t,t+1} (i) \cdot x_{n,t+1} (i) = .25 \cdot .5 \cdot 2 + .25 \cdot 1 \cdot 1 + .5 \cdot 1 \cdot 1 = 1
\]
We now use the fact that the expectation of the product of two variables is equal to the product of the expectations of each variable plus their covariance, \( E(YZ) = E(Y)E(Z) + \text{Cov}(EZ) \). Omitting time-subscripts for simplicity, the above can be rewritten as

\[
q_n = E(D) \cdot E(x_n) + \text{Cov}(D, x_n) = E(D) \left[ E(x_n) + \frac{\text{Cov}(D, x_n)}{E(D)} \right]
\]

The first term in the squared bracket is the expected cash flow from the asset, the second term is an ‘adjustment for risk.’

Work out the numerical example above. You should get \( ED = 0.875 \); \( Ex = 1.25 \); \( \text{Cov} = -0.09375 \).
Sure bond, risky asset

Consider a bond paying a non-stochastic cash flow \( x = 1 \) in all circumstances, so that \( \text{Cov}(D, 1) = 0 \). For this “sure bond”:

\[
q_{\text{sure bond}} = E(D) \cdot 1 \equiv \frac{1}{R} = \frac{1}{1 + r}
\]

The **rate of interest** is defined as the inverse of its price, that is, the inverse of the expected discount rate \( R = 1/E(D) \).

In our numerical example, \( 1 + r = 1/ED = 1/0.875 = 1.143 \), \( r \) is about 14%.

Using \( R \), rewrite the price of a generic asset \( n \) as

\[
q_n = E(D)E(x_n) + \text{Cov}(D, x_n) = \frac{E(x_n)}{R} + \text{Cov}(D, x_n) \tag{1}
\]

Given \( R \), \( q_n \) is increasing in

(1) the present value of the expected cash flow, discounted at the rate \( R \);
(2) the covariance between \( D \) and \( x \).
2. Understanding risk and return

$Cov > 0$ means that the cash flow from the asset tends to be systematically higher in situations — ‘states of the world’ — in which agents value resources more—i.e. $\mathcal{D}$ is high.

Understanding risk:

- For a given expected cash flow, an asset is riskier, the less it pays in circumstances in which additional resources are highly valued by the investors, that is, the lower $Cov (\mathcal{D}, x_n)$.
  - The higher the risk, the lower the price of the asset.
- Conversely: for a given expected cash flow, an asset is less risk, the higher $Cov (\mathcal{D}, x_n)$—implying a high price.

At least three key principles follow.
3. Three key principles

1. Higher risk must be compensated by higher expected return

Define the rate of return on the asset $R_n = \frac{x_n}{q_n}$. (note that $R_n$ varies stochastically with the cash flow ex post. Dividing (1) by $q_n$

$$1 = E_t \left( D_{t,t+1} \cdot \frac{x_{n,t+1}}{q_{n,t}} \right) \implies 1 = E(D) E(R_n) + \text{Cov}(D, R_n)$$

Define ‘excess return’ as $E(R_n) - R$:

$$R = \frac{E(x_n)}{q_n} + R \text{Cov} \left( D, \frac{x_n}{q_n} \right) = E(R_n) + R \text{Cov}(D, R_n) \Rightarrow$$

Excess return

$$E(R_n) - R = -R \text{Cov}(D, R_n) \leq 0$$

Keep in mind that the covariance on the right-hand side can have either sign: some assets may pay a negative excess-returns—reflecting the fact that they are less risky than a sure bond.
Three key principles

2. The volatility of returns needs not be compensated with higher expected returns

Consider an asset whose stochastic return happens to satisfy

\[ \text{Cov} (\mathcal{D}, R_n) = 0. \]

Its excess return \( E(R_n) - R = 0 \) is zero, regardless of \( \text{Var}(R_n) \). This asset will have the same price as a sure bond yielding \( R \), even if the variance of its return is extremely high.

Explain why and work out a numerical example.
Definition: If an investor’s discount factor $D$ is the same across all states of the world, so that $\text{Cov}(D, R_n) = 0$ for all assets, that investor is risk neutral.

3. Under risk neutrality, the price of any asset depends exclusively on the expected value of its cash flows:

$$q_n = \frac{E(x_n)}{R}.$$  

All excess returns are zero regardless of the stochastic properties of $R_n$

$$E(R_n) - R = 0.$$  

All returns are equalized ex ante in expectations (although they may and will be different ex post).
4. Discounting real and nominal cash flows

Let \( X_n \) and \( Q_n \) denote, respectively, the cash flow and the price of an asset in nominal terms (i.e. in units of currency). Define \( P \) as the price level, such that \( X_n = P \cdot x_n \) and \( Q_n = P \cdot q_n \). Then

\[
\frac{q_n}{Q_{n,t}} = E \left[ D_{t,t+1} \cdot \frac{x_n}{P_{t+1}} \right]
\]

Rearranging the above

\[
Q_{n,t} = E \left[ \left( D_{t,t+1} \cdot \frac{P_t}{P_{t+1}} \right) \cdot X_{n,t+1} \right]
\]

The stochastic discount factor for nominal cash flows is \( D_{t,t+1} \frac{P_t}{P_{t+1}} \).
Consider a bond paying $X = 1 + i$ units of domestic currency one period ahead, traded at the price $Q = 1$ today. Provided the nominal payoff is not random, we can write (omitting time-subscripts except for the price index):

$$1 = E \left[ \left( D \frac{P_t}{P_{t+1}} \right) (1 + i) \right] = \left\{ E \left( D \right) E \left( \frac{P_t}{P_{t+1}} \right) + \text{cov} \left[ \left( D, \frac{P_t}{P_{t+1}} \right) \right] \right\} (1 + i)$$

We can naturally think of $\frac{P_t}{P_{t+1}}$ as the inverse of one plus the rate of inflation (defined using the relevant consumer price index)

$$\frac{P_t}{P_{t+1}} = \frac{1}{1 + \pi_{t+1}}$$
Nominal and real interest rates

Divide through by both $E(D) = 1/(1 + r)$ and $E\left(\frac{P_t}{P_{t+1}}\right)$:

$$\frac{1}{1 + i} = \left\{ \left[ \frac{1}{1 + r} E\left(\frac{P_t}{P_{t+1}}\right) \right] + \text{cov}\left[\left(D, \frac{P_t}{P_{t+1}}\right)\right] \right\} = \left\{ \left[ \frac{1}{1 + r} E\left(\frac{1}{1 + \pi_{t+1}}\right) \right] + \text{cov}\left[\left(D, \frac{1}{1 + \pi_{t+1}}\right)\right] \right\}$$

- Under risk neutrality

$$(1 + i) = \frac{(1 + r)}{E\left(\frac{P_t}{P_{t+1}}\right)}$$
The real rate is often approximated taking logs, as follows

\[ i \approx r + E_t \pi_{t+1} \]

In approximation and under risk neutrality, the nominal rate of interest is the sum of the real rate and the expected rate of inflation.

Note the difference

\[ r \approx i - E_t \pi_{t+1} \quad \text{versus} \quad r \approx i - \pi_{t+1} \]

The latter is calculated by subtracting the \textit{realized} (not the expected) rate of inflation from the nominal interest rate.
Consider two ways to define the nominal exchange rate, labelled direct and indirect:

- the foreign currency is **base**, the domestic currency is **quoted** \(\Rightarrow\) denoted \(\mathcal{E}\)
  - Units of domestic currency per unit of foreign currency
  - An increase in \(\mathcal{E}\) is a depreciation of the domestic currency. The rate of depreciation between \(t\) and \(t+1\) is
    \[
    \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} - 1
    \]

- the foreign currency is **quoted**, the domestic currency is **base** \(\Rightarrow\) denoted \(\mathcal{S} = \frac{1}{\mathcal{E}}\)
  - An increase in \(\mathcal{S}\) is an appreciation of the domestic currency.

We will mostly use the direct, \(\mathcal{E}\).
Domestic and foreign currency denominated assets

Denote variables denominated in foreign currency with an asterisk ‘*’. So, an asset denominated in foreign currency will trade at the price $Q_n^*$, and deliver a cash flow $X_n^*$.

- For domestic investors trading a foreign currency denominated asset will have:
  \[ E_t \cdot Q_n^* = E \left( D \frac{P_t}{P_{t+1}} \left( E_{t+1} \cdot X_n^* \right) \right) \]

- For foreign investors:
  \[ Q_n^* = E \left( D^* \frac{P_t^*}{P_{t+1}^*} X_n^* \right) \]

- Domestic and foreign investors may differ in their discount factor and price indexes (also, they may have different beliefs/information, assessment of probabilities etc., so that we could write $E$ and $E^*$, but let’s abstract from these difference for now).

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Suppose financial markets are perfectly integrated across borders, so that the law of one price holds (each asset is traded at the same price everywhere). In equilibrium domestic and foreign investors value the cash flow from the asset in the same way:

\[
Q^*_n = E \left( D \frac{P_t}{P_{t+1}} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \cdot X^*_n \right) = E \left( D^* \frac{P^*_t}{P^*_{t+1}} \cdot X^*_n \right)
\]

Analogously, for an asset in domestic currency:

\[
Q_n = E \left( D \frac{P_t}{P_{t+1}} \cdot X_n \right) = E \left( D^* \frac{P^*_t}{P^*_{t+1}} \frac{\mathcal{E}_t}{\mathcal{E}_{t+1}} \cdot X_n \right)
\]

This set of conditions ties together discount rates, rates of inflation, rates of currency depreciation.
7. Portfolios, arbitrage portfolios and arbitrage opportunities

Consider an economy with $N$ assets. Suppose that at time $t + 1$ asset returns may vary randomly across $S$ states of nature ($s = 1, 2, ..., S$), each occurring with probability $\psi(s)$, $\sum_{s=1}^{S} \psi(s) = 1$.

Let $X$ denote the matrix displaying the cash flow from each asset by state of nature. The dimension is $S$ by $N$.

For instance, posit $S = 3$ and $N = 2$

\[ X = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \]
Portfolios

Consider an investor with wealth $W$, investing an amount $\omega_n$ in each of the $n$ assets. Let $\omega$ (in bold) denote the vector of investment in different assets, so that

$$\sum_n \omega_n = 1' \omega = W$$

where $1'$ is a row vector of 1 with dimension $n$. Note that $\omega$ can be positive or negative depending on whether the investor buys (goes long) or sells (goes short on) an asset.

In our example, let $W = 4$ and investment

$$\omega = [1 \quad 3]'$$

The cash flow from the portfolios of the two assets is a cash flow equal to

$$X\omega = \begin{bmatrix} 10 \\ 5 \\ 9 \end{bmatrix}.$$
Let $w_n$ denote portfolio shares, i.e. $w_n = \frac{\omega_n}{W}$. The investor’s wealth can be re-written as:

$$\sum_n \frac{\omega_n}{W} = 1' \frac{\omega}{W} = 1' w = 1$$

Let $R$ denote the matrix of returns from assets (obtained by dividing the cash flow by prices). The return on a portfolio is thus $R\omega$.

If the two assets have prices 1 and 2

$$R = \begin{bmatrix} 1 & 3/2 \\ 2 & 1/2 \\ 3 & 1 \end{bmatrix}$$
Arbitrage portfolios versus arbitrage opportunities

Define an **arbitrage portfolio** as a portfolio committing zero wealth:

$$1' \omega = 0$$

An instance is $\omega = [1 - 1]'$, the investor buys (goes long on) the first asset and sell (going short on) the second asset in equal amounts. In our numerical example, the cash flow from this portfolio is $[-2 \ 1 \ 1]'$: the cash is positive in states 2 and 3, but negative in state 1.

An **arbitrage opportunity** arises if investors can form an arbitrage portfolio (committing zero wealth today), that generate non-negative cash flows in all states of the world, positive in at least one state of the world:

$$1' \omega \leq 0 \quad \text{and} \quad X \omega \geq 0$$

where the second inequality has at least one strictly positive component.
In our example, an arbitrage opportunity would arise if, at the current asset prices, an agent can buy one asset and sell short a second one combining them in the following portfolio $\omega = [1 - 1]'$ and obtain:

\[
X = \begin{bmatrix}
1 & 1 \\
2 & 1 \\
3 & 2 \\
\end{bmatrix}
\]

Note that there is no risk involved—the investor uses no wealth at $t$ to form the portfolio ($1'\omega = 0$), and gets a positive cash flows in the 2nd and 3rd states of the world (0 in the first): $X\omega = 2$. The arbitrage opportunity produces is a ‘free lunch.’
If at the current asset prices there are arbitrage opportunities, investors will have a strong incentive to play the market as much as they can, i.e. in the previous example, they would like to invest $\omega = \lim_{a \to \infty} [+a, -a]$.

But this would ultimately raise (lower) the prices of the under(over)-valued asset, up to causing the arbitrage opportunity to disappear.

Note that arbitrage opportunities can also arise if

$$1' \omega < 0 \quad \text{and} \quad X\omega \geq 0$$

The investor obtains resources today without committing any resource tomorrow (since at $t+1$ positive and negative cash flows from the assets in the portfolios offset each other).