Appendix
Exchange Rate Misalignment, Capital Flows and Optimal Monetary Policy Trade-offs

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1 Appendix A

1.1 The transmission of monetary policy with imperfect capital markets

In this appendix, we analyze how monetary policy impacts the welfare-relevant gaps defined in the main text. As is well known, there are notable differences in the transmission of monetary decisions across LCP and PCP economies. Specifically, a monetary expansion causing nominal depreciation weakens the terms of trade under PCP but tends to strengthen the terms of trade under LCP. Here, our specific interest is to understand how monetary transmission is affected by financial distortions.

Starting with the LCP model, consider for simplicity a Home monetary shock such that CPI inflation follows an autoregressive process, \( a_H \pi_{H+t+s} + (1 - a_H) \pi_{F+t+s} = \rho^s \pi > 0, s \geq 0 \)—assuming that the Foreign monetary authority responds by keeping CPI price stability, i.e., \( a_H \pi_{F+t+s} + (1 - a_H) \pi_{H+t+s} = 0, s \geq 0 \). For the reasons explained in the text, we focus on the case \( \eta = 0 \), when the LCP model is relatively straightforward to solve. With \( \eta = 0 \), the responses of the key variables are given in Table A1. In the table, since an expansionary Home monetary policy shock is obviously inefficient (all first-best deviations are equal to zero), the responses of welfare-relevant gaps coincide with the response of macro variables.

<table>
<thead>
<tr>
<th>Table A1: The effect of a monetary policy shock under LCP</th>
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<tbody>
<tr>
<td>( \tilde{W}_{t+s} = \tilde{W}_t = \frac{(\sigma - 1)}{2(1 - a_H) + \sigma} \frac{a_H}{2 a_H ((\phi - 1) \frac{1 - \nu_1}{\beta_2} + 1)^{-1}} \frac{1 - \beta}{\alpha} \tilde{\pi} )</td>
</tr>
<tr>
<td>( \tilde{B}_t = (1 - a_H) \left{ 2a_H (\phi - 1) \frac{(\beta \nu_2 - 1)}{\beta_2} \frac{1}{1 - \beta \nu_2} \tilde{W}_t + \frac{(\sigma - 1)}{\sigma} \frac{(1 - \rho)}{\alpha} \beta \tilde{\pi} \right} )</td>
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<tr>
<td>( \tilde{T}_{t+s} + \Delta \tilde{t}<em>s = -\frac{1 - \nu^{s+1}}{1 - \nu_1} \frac{\beta \nu_2 - 1}{\beta_2} \tilde{W}</em>{t+s} )</td>
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<tr>
<td>( \tilde{\Delta}<em>{t+s} = \frac{(1 - \rho \beta)}{1 - \sigma \beta + 1} \rho^s \tilde{\pi} - (2a_H - 1) \left[ 1 - \frac{1 - \nu^{s+1}}{1 - \nu_1} \frac{(\beta \nu_2 - 1)}{\beta_2} \right] \tilde{W}</em>{t+s} )</td>
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<tr>
<td>( \tilde{Q}<em>{t+s} = \frac{(1 - \rho \beta)}{1 - \sigma \beta + 1} \rho^s \tilde{\pi} - (2a_H - 1) \tilde{W}</em>{t+s} )</td>
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<tr>
<td>( \sigma \tilde{Y}<em>{H,t+s} = a_H \frac{(1 - \rho \beta)}{1 - \sigma \beta + 1} \rho^s \tilde{\pi} - (1 - a_H) \left[ 1 + 2a_H \left( \frac{1 - \nu^{s+1}}{1 - \nu_1} \frac{(\beta \nu_2 - 1)}{\beta_2} - 1 \right) \right] \tilde{W}</em>{t+s} )</td>
</tr>
<tr>
<td>( \sigma \tilde{Y}<em>{F,t+s} = (1 - a_H) \frac{(1 - \rho \beta)}{1 - \sigma \beta + 1} \rho^s \tilde{\pi} + (1 - a_H) \left[ 1 + 2a_H \left( \frac{1 - \nu^{s+1}}{1 - \nu_1} \frac{(\beta \nu_2 - 1)}{\beta_2} - 1 \right) \right] \tilde{W}</em>{t+s} )</td>
</tr>
<tr>
<td>( \sigma \tilde{D}<em>{t+s} = \frac{(1 - \rho \beta)}{1 - \sigma \beta + 1} \rho^s \tilde{\pi} + 2 (1 - a_H) \tilde{W}</em>{t+s} )</td>
</tr>
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</table>

When markets are incomplete, a monetary shock generally causes the wealth gap \( \tilde{W}_t \) to deviate from zero (recall that in the bond economy \( E_t \tilde{W}_{t+1} = \tilde{W}_t \))—implying that the effects of a monetary policy shock under incomplete markets are generally different than those under complete markets. In a few notable special cases, however, the effects of monetary policy are the same as in economics with complete markets. One such case is \( \sigma = 1 \) (log consumption utility), whereas \( \tilde{W}_t = 0 \), and neither capital flows \( \tilde{B}_t \), nor the relative price misalignment, \( \tilde{T}_t + \Delta_t \), are affected by monetary policy. In this special case, a monetary easing unambiguously results in positive domestic and foreign output gaps, a
positive real exchange rate gap, and a higher relative demand gap. Relative to this benchmark, if the gap $W_t$ is positive the effects of monetary policy on the domestic output and the real exchange rate gaps are smaller, while the foreign output and the relative demand gaps react more. These differences reflect the fact that the misalignment of output and the relative demand gaps react more. These differences reflect the fact that the misalignment $\Delta y_t + \Delta D_t$ is negative when $W_t > 0$, implying “expenditure switching” in favor of foreign exports. The opposite is true if the wedge is negative: the domestic output and real exchange rate gaps react by more, while the transmission abroad is muted.

A monetary expansion can open a wealth gap in different directions, depending on elasticities. To see this, consider the following threshold expressed in terms of the trade elasticity:

$$\phi > 1 - \frac{2a_H - 1 + \frac{2(1-s_H)}{\sigma}}{2a_H \frac{\phi - 1}{1-\nu_1}} \geq 0.$$ 

A monetary easing brings about a positive gap $\tilde{W}_t$, either when $\sigma > 1$ and $\phi$ is above the threshold shown above; or when $\sigma < 1$ and $\phi$ is below the threshold.

By the same token, a monetary expansion can lead to either an external surplus or an external deficit. From the second equation in the table, a sufficient condition for a monetary easing to lead to an inefficient capital outflow $\tilde{B}_t = \tilde{B}_t > 0$ is that both $\sigma > 1$ and $\phi > 1$— recall that $\tilde{B}_t^b = 0$. A sufficient condition for inflows, $\tilde{B}_t < 0$, instead, is $\sigma < 1$ and $\phi$ below the threshold. It follows that, depending on parameter values, a positive gap $\tilde{W}_t > 0$ brought about by a monetary expansion may be associated with either outflows or inflows of capital. These in turn would attenuate (or amplify) the effects of monetary policy on domestic output and the real exchange rate (domestic consumption and foreign output).

The transmission of monetary policy under PCP is shown in Table A2. Relative to the previous table, monetary easing is now modelled as an increase in domestic PPI inflation $\pi_{PPI} = \rho^* \pi > 0$, $s \geq 0$, again under the assumption that the Foreign monetary authority responds by keeping PPI price stability, i.e., $\pi_{PPI} = 0$, $s \geq 0$ and $\eta = 0$.

**Table A2: The effect of a monetary policy shock under PCP**

\[
\begin{align*}
\tilde{W}_{t+s} &= \tilde{W}_t = \frac{2a_H - 1 + \frac{2(1-s_H)}{\sigma}}{2a_H \frac{\phi - 1}{1-\nu_1}} \pi \\
\tilde{B}_t &= (1-a_H) \left(\frac{2a_H - 1}{1 - \alpha H + 1 - \beta H} \right) \frac{1}{\alpha} \pi \\
\tilde{Q}_{t+s} &= (2a_H - 1) \tilde{T}_{t+s} = (2a_H - 1) \left(1 - a_H \phi - 1\right) \rho^* \pi - \tilde{W}_{t+s} \\
\sigma \tilde{Y}_{H, t+s} &= [1 + 2a_H (1 - a_H) (\sigma \phi - 1)] \frac{1}{1 - \alpha H + 1 - \beta H} \rho^* \pi - (1 - a_H) 2a_H (\sigma \phi - 1) + 1 \tilde{W}_{t+s} \\
\sigma \tilde{Y}_{PPI, t+s} &= -2a_H (1 - a_H) (\sigma \phi - 1) \frac{1}{1 - \alpha H + 1 - \beta H} \rho^* \pi + (1 - a_H) 2a_H (\sigma \phi - 1) + 1 \tilde{W}_{t+s} \\
\sigma \tilde{D}_{t+s} &= \frac{(2a_H - 1)}{1 - \alpha H + 1 - \beta H} \rho^* \pi + 2(1 - a_H) \tilde{W}_{t+s}.
\end{align*}
\]

An expansionary Home monetary policy shock also causes the gap $\tilde{W}_t$ to deviate from zero under PCP: under incomplete markets, the effects of a monetary policy
shock do not coincide with those under complete markets. Again there are a few notable exceptions: under PCP, the special case in which monetary policy affects neither $\tilde{W}_t (= 0)$ nor capital flows arises when $\phi = \frac{1 + 2a_H}{2a_H}$; if $\sigma = 1$, then, this requires $\phi = 1$—a Cobb-Douglas consumption aggregator. In this special case, just like under complete markets, a monetary easing unambiguously results in a higher domestic output, relative demand and real exchange rate gaps. However, foreign output is affected only when $\sigma \phi \neq 1$, and increases if $\sigma \phi < 1$, namely, when goods are Edgeworth-complement. Relative to the benchmark with $\phi = \frac{1 + 2a_H}{2a_H}$, similar to LCP, a positive (negative) wealth gap means that the effects of monetary policy on domestic output and the real exchange rate are smaller (larger) than under complete markets, while domestic consumption and foreign output react more (less). These effects reflect the fact that the response of the terms of trade, $\tilde{E}_t$, is also smaller (larger), implying a weaker (stronger) expenditure switching in favor of Home goods. Specifically, the wealth gap is positive when the following conditions hold:

$$\phi > \frac{1 + 2a_H}{2a_H}, \quad \phi < \frac{1 - 2(1-a_H)}{2a_H}.$$  

From the second equation in the table, it is apparent that, for a monetary easing to lead to an inefficient capital outflow on impact, $\tilde{B}_t > 0$, it must be the case that $\phi > \frac{1 + 2a_H}{2a_H}$. Otherwise, it leads to capital inflows. Therefore, also under PCP a positive $\tilde{W}_t > 0$ may be associated with either outflows or inflows of capital, in turn attenuating or amplifying the effects of monetary policy on domestic output and the real exchange rate (domestic consumption and foreign output).

### 1.2 Costly intermediation and stationarity of net foreign assets

Our results so far have been derived in a specification of the model in which both $\tilde{B}_t$ and $\tilde{W}_t$ are not stationary. In this subsection, we show that nonstationarity does not play any substantive role. In the literature, a standard approach to ensure that $\tilde{B}_t$ is stationary in bond economies is to assume that its changes are subject to some (portfolio) adjustment costs; Gabaix and Maggiori [2015] have recently shown that this sluggish adjustment can result from costly intermediation of cross-border flows when financial intermediaries operate under borrowing constraints. In our framework, a simple way to capture the same idea is to posit deviations from the uncovered interest rate parity condition that are proportional to net foreign assets:

$$E_t \tilde{W}_{t+1} - \tilde{W}_t = -\delta \tilde{B}_t.$$  

With this modification, the solutions for $\tilde{B}_t$ and $\tilde{W}_t$ in the CO economy become:

$$\tilde{B}_t = \gamma_1 \tilde{B}_0 + (1 - a_H) \sum_{j=0}^{\infty} \gamma_2^{-j-1} E_t \left[ (\tilde{\zeta}_{C,t+1+j} - \tilde{\zeta}_{C,t+1+j}) - (\tilde{\zeta}_{C,t+j} - \tilde{\zeta}_{C,t+j}) \right],$$

4
\[
\tilde{W}_t = \left( \frac{\tilde{B}_{t-1} - \beta \tilde{B}_t}{1 - a_H} \right) - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^* \right) \\
= - \left[ (\tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^*) + \sum_{j=0}^{\infty} \gamma_2^{-j-1} E_t \left[ (\tilde{\zeta}_{C,t+1+j} - \tilde{\zeta}_{C,t+1+j}^*) - (\tilde{\zeta}_{C,t+j} - \tilde{\zeta}_{C,t+j}^*) \right] - \frac{\gamma_1 - \beta}{1 - a_H} \tilde{B}_{t-1} \right].
\]

where \( \beta < \gamma_1 < 1 < \gamma_2 \) are the roots of the characteristic equation associated with the above second-order difference equation:

\[
\beta \gamma_2 - (1 + \beta + \beta \delta) \gamma + 1 = 0.
\]

Both \( \tilde{W}_t \) and \( \tilde{B}_t \) are now stationary, but still functions of exogenous shocks only, so the optimal targeting rules are the same as those derived above under both LCP and PCP. Therefore, optimal monetary policy will react in the same way to a capital inflow, by tightening under LCP and easing under PCP (although of course with a different strength). Clearly, setting \( \delta = 0 \) in the last expression leads to \( \gamma_1 = 1 \) and \( \gamma_2 = 1/\beta \), which yields expressions (??) and (??) above.

### 1.3 Determinants of the optimal Home monetary stance under PCP with a non-unitary elasticity

To prove the result in Subsection 5.2 in the text, we start noting that, for low enough trade elasticities, the impact response of relative consumption to capital inflows,

\[
\tilde{C}_t = \underbrace{2 (1 - a_H) \tilde{W}_t}_{\text{sign}(-\tilde{B}_t)} \left\{ \frac{2 a_H (\phi - 1) + 1 + (2 a_H - 1)}{\beta \kappa_2} \right\},
\]

switches sign and becomes negative. This will be the case for values of \( \phi \) below the threshold

\[
\phi \leq \frac{(2 a_H - 1)(\beta \kappa_2 - 1)}{2 a_H} \leq \frac{(2 a_H - 1)}{2 a_H} < 1.
\]

where an expansionary monetary policy stance is motivated by an inefficiently low domestic demand. The monetary boosts causes the output gap to become consistently positive. On impact, the optimal response can be written as follows

\[
\tilde{Y}_{H,t} = -\underbrace{(1 - a_H) \tilde{W}_t}_{\text{sign}(\tilde{B}_t)} \left\{ \frac{1}{\beta \kappa_2} \left\{ (\beta \kappa_2 - 1) [2 a_H (\phi - 1) + 1] + 1 \right\} \right\},
\]

where the term outside the curly brackets has the same sign as capital inflows \( \tilde{B}_t \). Therefore, whether the optimal policy turns the output gap positive or negative depends on the sign of the term in curly brackets, in turn a function of \( \phi \). For \( \phi < 1 \), the second term in the curly brackets \((4 a_H^2 \phi (\phi - 1)) \) is always negative, and converges to zero as \( \phi \to 0 \). The first term is a square and thus
always positive, but converges to zero as \( \phi \) converges to the cutoff point \( \frac{2a_H - 1}{2a_H} \) from above; it then becomes increasingly positive for lower values of elasticities. This implies that, under the optimally expansionary monetary policy, there is a range of elasticities around the cutoff point for which the output gap is positive. This range becomes larger, the closer \( \beta x_2 \) is to 1, i.e., the stickier prices are, since the first term in curly brackets goes to zero. Importantly, this is in contrast to the natural rate allocation in which the output gap is negative for any value of the elasticity (see Table 5).

An expansionary stance dictated by concerns with domestic stabilization of course exacerbates the real misalignment. Under the optimal policy, the response of the terms-of-trade gap is

\[
\tilde{T}_t = \frac{2Y_{H,t} - (2a_H - 1) \tilde{W}_t}{4a_H (1 - a_H) (\phi - 1) + 1} - \frac{\tilde{W}_t}{2a_H (\phi - 1) + 1} \cdot \left\{ \frac{1}{\beta x_2} \left[ \frac{2a_H (\phi - 1) + 1}{\beta x_2} \right] \right\},
\]

where the term in curly brackets is positive if

\[
\phi \geq \frac{1 + (2a_H - 1)(\beta x_2 - 1)}{\beta x_2} < 1,
\]

and is negative if \( \phi \leq \frac{2a_H - 1}{2a_H} \). It follows that, for \( \phi \leq \frac{2a_H - 1}{2a_H} \), the optimal Home monetary expansion causes the terms of trade to be excessively weak.
2 Appendix B

In this appendix we first derive the quadratic loss function under LCP and generically incomplete markets. The PCP case can be understood as a special case where law of one price (LOOP) deviations are set to zero. Using this loss function, in Section 2, we characterize the constrained efficient allocation and targeting rules.

2.1 Deriving quadratic loss function under LCP and generically incomplete markets

In this section of the appendix we derive the quadratic loss function under LCP and generically incomplete markets. The PCP case can be understood as a special case where law of one price (LOOP) deviations are set to zero.

Write the one-period utility flow:

\[ U(C) - V(L) = \zeta_C C^{1-\sigma} - \frac{L^1}{1+\eta} \]

Under the assumption of an efficient steady state with subsidy \( \frac{(\theta - 1)(1 - \tau)}{\theta} = 1 \), so that \( U'(C) = -V'(L) \), the second order approximation of utility is as follows:

\[ \hat{C}_t - \hat{Y}_{H,t} + \left( \frac{1-\sigma}{2} \hat{C}_t + \hat{\zeta}_{C,t} \right) \hat{C}_t - (1 + \eta) \left( \frac{1}{2} \hat{Y}_{H,t} - \hat{\zeta}_{Y,t} \right) \hat{Y}_{H,t} + \]

\[ -\frac{1}{2} \frac{\theta \alpha}{(1 - \alpha \beta)(1 - \alpha)} \left[ a_H \pi_{H,t}^2 + (1 - a_H) \pi_{H,t}^2 \right] + t.i.p. + o(\varepsilon^3) \]

where we have used the log-linear approximation to the aggregate production function: \( \hat{Y}_{H,t} = \hat{\zeta}_{Y,t} + \hat{L}_t \). Inflation rates appear in this expression because the second order approximation of labor effort is proportional to price dispersion, which in turn is a function of sectoral inflation rates under LCP and Calvo price-setting with symmetric probabilities \( \alpha \) (see Engel (2009)).

Similarly, for the Foreign country we have,

\[ \hat{C}_{t}^* - \hat{Y}_{F,t} + \left( \frac{1-\sigma}{2} \hat{C}_{t}^* + \hat{\zeta}_{C^*,t} \right) \hat{C}_{t}^* - (1 + \eta) \left( \frac{1}{2} \hat{Y}_{F,t} - \hat{\zeta}_{Y,t}^* \right) \hat{Y}_{F,t} + \]

\[ -\frac{1}{2} \frac{\theta \alpha}{(1 - \alpha \beta)(1 - \alpha)} \left[ a_H \pi_{F,t}^2 + (1 - a_H) \pi_{F,t}^2 \right] + t.i.p. + o(\varepsilon^3) \]

Under cooperation, the global policy objective function \( \mathcal{L}_t^W \) will be the sum of the two country-specific terms.
The first order approximations of aggregate demand yields,

\[ \mathcal{L}_W^t = (\tilde{C}_t + \tilde{C}_t^*) - (\tilde{Y}_{H,t} + \tilde{Y}_{F,t}) + \left( \frac{1 - \sigma}{2} (\tilde{C}_t + \tilde{\zeta}_{C,t}) \right) \tilde{C}_t + \left( \frac{1 - \sigma}{2} \tilde{C}_t^* + \tilde{\zeta}_{C,t}^* \right) \tilde{C}_t^* \]

\[ - (1 + \eta) \left( \frac{1}{2} \tilde{Y}_{H,t} - \tilde{\zeta}_{Y,t} \right) \tilde{Y}_{H,t} - (1 + \eta) \left( \frac{1}{2} \tilde{Y}_{F,t} - \tilde{\zeta}_{Y,t} \right) \tilde{Y}_{F,t} + \]

\[ - \frac{1}{2} \frac{\theta \alpha}{1 - \alpha} \left( \left[ a_H \pi_H^2 + (1 - a_H) \pi_H^2 \right] + \left[ a_H \pi_F^2 + (1 - a_H) \pi_F^2 \right] \right) + t.i.p. + o \left( \varepsilon^3 \right), \]

The objective of this appendix is to rewrite the above as a quadratic loss function in terms of gaps and misalignments.

### 2.1.1 Useful relationships

We begin by writing some useful relations. The real exchange rate is related to the terms of trade and deviations from the law of one price as follows:

\[ \tilde{Q}_t = (2a_H - 1) \tilde{T}_t + 2a_H \tilde{\Delta}_t. \]  

(1)

The first order approximations of \( \tilde{C}_t \) and \( \tilde{C}_t^* \), are given by,

\[ \tilde{C}_t^* = \tilde{C}_t - \sigma^{-1} \left[ \tilde{Q}_t + \tilde{W}_t + \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^* \right) \right] \]  

(2)

\[ \tilde{C}_t = \frac{1}{2} \left\{ \tilde{Y}_{H,t} + \tilde{Y}_{F,t} + \sigma^{-1} \left[ \tilde{Q}_t + \tilde{W}_t + \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^* \right) \right] \right\}. \]

The first order approximations of \( \tilde{C}_t \) and \( \tilde{C}_t^* \) imply,

\[ -(\tilde{C}_t - \tilde{Y}_{H,t}) = \tilde{C}_t^* - \tilde{Y}_{F,t} = \frac{1}{2} \left\{ \tilde{Y}_{H,t} - \tilde{Y}_{F,t} - \sigma^{-1} \left[ \tilde{Q}_t + \tilde{W}_t + \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^* \right) \right] \right\}. \]

(3)

The first order approximation of aggregate demand yields,

\[ \tilde{C}_t = \tilde{Y}_{H,t} - (1 - a_H) \sigma^{-1} \left[ \sigma \phi \tilde{T}_t + (\sigma \phi - 1) \tilde{Q}_t - \tilde{W}_t - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^* \right) \right] \]

\[ \tilde{C}_t^* = \tilde{Y}_{F,t} + (1 - a_H) \sigma^{-1} \left[ \sigma \phi \tilde{T}_t + (\sigma \phi - 1) \tilde{Q}_t - \tilde{W}_t - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^* \right) \right] \]

Combining the first order approximations of aggregate demand, we obtain,

\[ \tilde{C}_t = \tilde{Y}_{H,t} - \frac{1 - a_H}{\sigma} \left[ 2a_H \phi \sigma \left( \tilde{T}_t + \tilde{\Delta}_t \right) - \tilde{Q}_t - \tilde{W}_t - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^* \right) \right]. \]

Combining the two expressions for consumption, we obtain the following expression for the terms of trade:

\[ |4a_H (1 - a_H) (\sigma \phi - 1) + 1| (\tilde{T}_t + \tilde{\Delta}_t) = \]  

\[ \sigma \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) - (2a_H - 1) \left[ \tilde{W}_t + \tilde{\Delta}_t + \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^* \right) \right] \]  

(4)
In addition, shocks can be expressed in terms of efficient output and the terms of trade,

\[ \hat{\zeta}_{C,t} + (1 + \eta) \hat{\zeta}_{Y,t} = \]

\[ (\eta + \sigma) \hat{Y}_{H,t}^{fb} - [2a_H (1 - a_H) (\sigma \phi - 1)] \left( \hat{T}_t^{fb} \right) + (1 - a_H) \left( \hat{\zeta}_{C,t} - \hat{\zeta}_{C,t}^* \right) \]

Next, using the first order approximation for domestic consumption, we can rewrite domestic marginal costs as follows,

\[ \sigma \hat{C}_t - \hat{\zeta}_{C,t} + \eta \hat{Y}_{H,t} - (1 + \eta) \hat{\zeta}_{Y,t} + (1 - a_H) \left( \hat{T}_t + \Delta_t \right) = \]

\[ (\eta + \sigma) \left( \hat{Y}_{H,t} - \hat{Y}_{H,t}^{fb} \right) + \]

\[ - (1 - a_H) \left[ (\sigma \phi - 1) \left( \hat{T}_t - \hat{T}_t^{fb} + \hat{Q}_t - \hat{Q}_t^{fb} \right) - \hat{W}_t - \Delta_t \right] \]

Rearranging,

\[ \frac{\sigma}{2} \hat{C}_t - \hat{\zeta}_{C,t} + \frac{\eta}{2} \hat{Y}_{H,t} - (1 + \eta) \hat{\zeta}_{Y,t} + \frac{1}{2} (1 - a_H) \left( \hat{T}_t + \Delta_t \right) = \]

\[ (\eta + \sigma) \left( \frac{1}{2} \hat{Y}_{H,t} - \hat{Y}_{H,t}^{fb} \right) - 2a_H (1 - a_H) (\sigma \phi - 1) \left( \frac{1}{2} \left( \hat{T}_t + \Delta_t \right) - \hat{T}_t^{fb} \right) + \]

\[ \frac{1}{2} (1 - a_H) \left( \hat{W}_t + \Delta_t - \left( \hat{\zeta}_{C,t} - \hat{\zeta}_{C,t}^* \right) \right) \]

### 2.1.2 The global loss function in terms of gaps and misalignments

To eliminate the linear terms from \( L_W \), we proceed as follows. First, we derive a second-order accurate expression for the sum of consumption across countries (the world aggregate demand) by summing up the budget constraints under LCP:

\[ \frac{P_H}{P} (C_H + C_H^*) + \frac{SP_F^*}{P} (C_F + C_F^*) = \frac{P_H}{P} Y_H + \frac{SP_F^*}{P} Y_F \]

\[ C + QC^* + \left( \frac{SP_H^*}{P_H} - 1 \right) \frac{P_F}{P} C_F - \left( \frac{SP_F^*}{P_H} - 1 \right) \frac{P_H}{P} C_H^* = \frac{P_H}{P} Y_H + \frac{P_F}{P} \frac{SP_F^*}{P} Y_F \]

\[ C + QC^* + (1 - a_H) \left[ (\Delta_F - 1) \left( \frac{P_F}{P} \right)^{1-\phi} C + (\Delta_H^{-1} - 1) \left( \frac{P_H}{P} \right)^{1-\phi} QC^* \right] = \]

\[ \frac{P_H}{P} Y_H + \frac{P_F}{P} \frac{SP_F^*}{P} Y_F \]

\[ C + QC^* + (1 - a_H) \left\{ \begin{array}{c}
(\Delta_F - 1) a_H T^{-\phi} \Delta_H^{\phi-1} + (1 - a_H) \left\{ \begin{array}{c}
C + \left( \frac{P_H}{P} \right)^{1-\phi} QC^*
\end{array} \right\} \\
(\Delta_H^{-1} - 1) a_H T^{1-\phi} \Delta_H^{1-\phi} + (1 - a_H) \left\{ \begin{array}{c}
QC^*
\end{array} \right\}
\end{array} \right\} = \]

\[ \left[ a_H + (1 - a_H) T^{1-\phi} \Delta_H^{1-\phi} \right]^{-\frac{1}{1-\phi}} Y_H + \]

\[ \left[ a_H + (1 - a_H) T^{\phi-1} \Delta_F^{\phi-1} \right]^{-\frac{1}{\phi}} Y_F. \]
The accurate second-order expression for the world demand is:

\[
\begin{aligned}
&\bar{C}_t + \bar{C}_t^* + \frac{1}{2} \left( \bar{C}_t^2 + \bar{C}_t^{*2} \right) + \bar{Q}_t + \frac{1}{2} \bar{Q}_t^2 + \bar{Q}_t \bar{C}_t^* + \\
&(1 - a_H) \left[ \Delta_{F,t} + \frac{1}{2} \Delta_{F,t}^2 + \Delta_{F,t} (\bar{C}_t + a_H (1 - \phi) \left( \bar{F}_t + \Delta_{H,t} \right) ) - \\
&\left( \Delta_{H,t} + \frac{1}{2} \Delta_{H,t}^2 \right) + \Delta_{H,t} - \Delta_{H,t} \left( \bar{C}_t^* + \bar{Q}_t - a_H (1 - \phi) \left( \bar{F}_t + \Delta_{F,t} \right) \right) \right] \\
=& \bar{Y}_{H,t} + \bar{Y}_{F,t} + \frac{1}{2} \left( \bar{Y}_{H,t}^2 + \bar{Y}_{F,t}^2 \right) - (1 - a_H) \left[ \bar{F}_t + \Delta_{H,t} + \frac{1}{2} \left( \bar{F}_t^2 + \Delta_{H,t}^2 \right) \right] - \\
&(1 - a_H) \left( \bar{Y}_{H,t} + \Delta_{H,t} \right) (1 - a_H) \left[ \phi - 1 + (1 - a_H) (1 - \phi) \left( \frac{1}{1 - \phi} + 1 \right) \right] \bar{F}_t \Delta_{H,t} + \\
&\frac{1}{2} (1 - a_H) \left[ \phi + (1 - a_H) (1 - \phi) \left( \frac{1}{1 - \phi} + 1 \right) \right] \left( \bar{F}_t^2 + \Delta_{H,t}^2 \right) + \\
&(1 - a_H) \left( \bar{F}_t + \Delta_{F,t} + \frac{1}{2} \left( \bar{F}_t^2 + \Delta_{F,t}^2 \right) \right) + \bar{Q}_t + \frac{1}{2} \bar{Q}_t^2 + \bar{Y}_{F,t} \bar{Q}_t + (1 - a_H) \bar{Y}_{F,t} \left( \bar{F}_t + \Delta_{F,t} \right) + \\
&(1 - a_H) \left( \bar{F}_t + \Delta_{F,t} \right) \bar{Q}_t + (1 - a_H) \left[ \phi - 1 + (1 - a_H) (1 - \phi) \left( \frac{1}{1 - \phi} + 1 \right) \right] \bar{F}_t \Delta_{F,t} + \\
&\frac{1}{2} (1 - a_H) \left[ (1 - a_H) \left( \frac{1}{1 - \phi} + 1 \right) (1 - \phi) + \phi - 2 \right] \left( \bar{F}_t^2 + \Delta_{F,t}^2 \right).
\end{aligned}
\]

As the linear terms in relative prices cancel out and under the maintained assumption of symmetry \( \Delta_{H,t} = \Delta_{F,t} = \Delta_t \), we get:

\[
\bar{C}_t + \bar{C}_t^* + \frac{1}{2} \left( \bar{C}_t^2 + \bar{C}_t^{*2} \right) + (1 - a_H) \left( \bar{C}_t - \bar{C}_t^* - \bar{Q}_t \right) \Delta_t =
\]

\[
\bar{Y}_{H,t} + \bar{Y}_{F,t} + \frac{1}{2} \left( \bar{Y}_{H,t}^2 + \bar{Y}_{F,t}^2 \right) + \left( \bar{Y}_{F,t} - \bar{C}_t^* \right) \bar{Q}_t + \\
(1 - a_H) \left( \bar{Y}_{F,t} - \bar{Y}_{H,t} \right) \left( \bar{F}_t + \Delta_t \right) + a_H (1 - a_H) \left( \phi \left( \bar{F}_t + \Delta_t \right) \right)^2 + \\
(1 - a_H) \left( 1 - 2a_H (1 - \phi) \right) \left( \bar{F}_t - 2a_H (1 - \phi) \Delta_t \right) \Delta_t,
\]

Second, we substitute in the approximation to the sum of consumption—in addition, we subtract \( \frac{1}{2} (1 - a_H) \bar{F}_t \left( \bar{Y}_{H,t} - \bar{Y}_{F,t} \right) \left( \frac{1}{2} \bar{C}_t - \bar{C}_t^* \right) \bar{Y}_{H,t} \) and \( \frac{1}{2} \left( \bar{C}_t^* - \bar{C}_t^* \right) \bar{Y}_{F,t} \) in order to have a second-order term in the product of output and marginal costs.
for each country.

\[
\mathcal{L}^W_t \times \hat{C}_t + \hat{C}_t^* - \hat{Y}_{H,t} - \hat{Y}_{F,t} + \left( \frac{1 - \sigma}{2} \hat{C}_t + \hat{\zeta}_{C,t} \right) \hat{C}_t + \left( \frac{1 - \sigma}{2} \hat{C}_t^* + \hat{\zeta}_{C,t}^* \right) \hat{C}_t^* - (1 + \eta) \left( \frac{1}{2} \hat{\gamma}_{Y,t} - \hat{\zeta}_{Y,t} \right) \hat{Y}_{H,t} - (1 + \eta) \left( \frac{1}{2} \hat{\gamma}_{Y,t} - \hat{\zeta}_{Y,t} \right) \hat{Y}_{F,t} - \\
\frac{1}{2 (1 - \alpha \beta) (1 - \alpha)} \left[ a_H \pi_{H,t}^2 + (1 - a_H) \pi_{H,t}^2 + a_H \pi_{F,t}^2 + (1 - a_H) \pi_{F,t}^2 \right] + t.i.p. + o(\varepsilon^3) \\
- \left( \frac{\sigma}{2} \hat{C}_t - \hat{\zeta}_{C,t} \right) \hat{C}_t - \left( \frac{\sigma}{2} \hat{C}_t^* - \hat{\zeta}_{C,t}^* \right) \hat{C}_t^* + \left( \hat{Y}_{F,t} - \hat{C}_t^* \right) \hat{Q}_t - (1 - a_H) \left( \hat{C}_t - \hat{C}_t^* - \hat{Q}_t \right) \hat{\Delta}_t - \\
\frac{1}{2} \left( 1 - a_H \right) \left( \hat{Y}_{H,t} - \hat{Y}_{F,t} \right) \left( \hat{T}_t + \hat{\Delta}_t \right) + (1 - a_H) a_H \phi \left( \hat{T}_t + \hat{\Delta}_t \right)^2 + (1 - a_H) \left( \left[ 1 - 2a_H (1 - \phi) \right] \hat{T}_t - 2a_H (1 - \phi) \hat{\Delta}_t \right) \hat{\Delta}_t - \\
\left( \frac{\eta}{2} \hat{\gamma}_{Y,t} - (1 + \eta) \hat{\zeta}_{Y,t} + \frac{1}{2} (1 - a_H) \left( \hat{T}_t + \hat{\Delta}_t \right) \right) \hat{Y}_{H,t} - \\
\left( \frac{\eta}{2} \hat{\gamma}_{Y,t} - (1 + \eta) \hat{\zeta}_{Y,t} - \frac{1}{2} (1 - a_H) \left( \hat{T}_t + \hat{\Delta}_t \right) \right) \hat{Y}_{F,t} - \\
\frac{1}{2 (1 - \alpha \beta) (1 - \alpha)} \left[ a_H \pi_{H,t}^2 + (1 - a_H) \pi_{H,t}^2 + a_H \pi_{F,t}^2 + (1 - a_H) \pi_{F,t}^2 \right] + t.i.p. + o(\varepsilon^3) \\
\end{array}
\]

\[
= - \left( \frac{\sigma}{2} \hat{C}_t - \hat{\zeta}_{C,t} \right) \left( \hat{C}_t - \hat{Y}_{H,t} \right) - \left( \frac{\sigma}{2} \hat{C}_t^* - \hat{\zeta}_{C,t}^* \right) \left( \hat{C}_t^* - \hat{Y}_{F,t} \right) - (1 - a_H) \left( \hat{C}_t - \hat{C}_t^* - \hat{Q}_t \right) \hat{\Delta}_t - \\
\left( \frac{\sigma}{2} \hat{C}_t - \hat{\zeta}_{C,t} + \eta \hat{\gamma}_{Y,t} - (1 + \eta) \hat{\zeta}_{Y,t} + \frac{1}{2} (1 - a_H) \left( \hat{T}_t + \hat{\Delta}_t \right) \right) \hat{Y}_{H,t} - \\
\left( \frac{\sigma}{2} \hat{C}_t^* - \hat{\zeta}_{C,t}^* + \eta \hat{\gamma}_{Y,t} - (1 + \eta) \hat{\zeta}_{Y,t}^* - \frac{1}{2} (1 - a_H) \left( \hat{T}_t + \hat{\Delta}_t \right) \right) \hat{Y}_{F,t} - \\
\frac{1}{2} \left( 1 - a_H \right) \left( \hat{T}_t + \hat{\Delta}_t \right) \left( \hat{Y}_{H,t} - \hat{Y}_{F,t} \right) + \\
(1 - a_H) a_H \phi \left( \hat{T}_t + \hat{\Delta}_t \right)^2 + (1 - a_H) \left( \left[ 1 - 2a_H (1 - \phi) \right] \hat{T}_t - 2a_H (1 - \phi) \hat{\Delta}_t \right) \hat{\Delta}_t - \\
\frac{1}{2 (1 - \alpha \beta) (1 - \alpha)} \left[ a_H \pi_{H,t}^2 + (1 - a_H) \pi_{H,t}^2 + a_H \pi_{F,t}^2 + (1 - a_H) \pi_{F,t}^2 \right] + t.i.p. + o(\varepsilon^3) .
\]

Some more substitutions and algebra follows. Using the expressions for shocks (5) and domestic marginal costs (6) in terms of efficient output and
terms of trade, we can express the loss in terms of output gaps, relative price misalignment, including \( \Delta_t \), and demand imbalances:

\[
\mathcal{L}_t^W \times - \left( \frac{\sigma}{2} \tilde{C}_t - \tilde{C}_{C,t} \right) \left( \bar{C}_t - \bar{Y}_{H,t} \right) - \left( \frac{\sigma}{2} \tilde{C}_t^* - \tilde{C}_{C,t}^* + \bar{Q}_t \right) \left( \tilde{C}_t^* - \tilde{Y}_{F,t} \right) -
\]

\[
(1 - a_H) \left( \bar{C}_t - \bar{C}_t^* - \bar{Q}_t \right) \Delta_t -
\]

\[
\left[ (\eta + \sigma) \left( \frac{1}{2} \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) - 2a_H (1 - a_H) (\sigma \phi - 1) \left( \frac{1}{2} \tilde{T}_t + \tilde{\Delta}_t \right) - \tilde{T}_t^{fb} \right] \tilde{Y}_{H,t} + -
\]

\[
\left[ (\eta + \sigma) \left( \frac{1}{2} \tilde{Y}_{F,t} - \tilde{Y}_{F,t} \right) + 2a_H (1 - a_H) (\sigma \phi - 1) \left( \frac{1}{2} \tilde{T}_t + \tilde{\Delta}_t \right) - \tilde{T}_t^{fb} \right] \tilde{Y}_{F,t} + -
\]

\[
\frac{1}{2} (1 - a_H) \left[ (\tilde{T}_t + \tilde{\Delta}_t) + \tilde{W}_t + \tilde{\Delta}_t - \tilde{C}_{C,t} - \tilde{C}_{C,t}^* \right] \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) +
\]

\[
(1 - a_H) a_H \phi \left( \tilde{T}_t + \tilde{\Delta}_t \right)^2 + (1 - a_H) \left( [1 - 2a_H (1 - \phi)] \tilde{T}_t - 2a_H (1 - \phi) \tilde{\Delta}_t \right) \tilde{\Delta}_t -
\]

\[
\frac{1}{2} \left( 1 - \alpha \beta \right) \left( 1 - \alpha \right) \left[ a_H \pi_{H,t}^2 + (1 - a_H) \pi_{H,t}^2 + a_H \pi_{F,t}^2 + (1 - a_H) \pi_{F,t}^2 \right] + t.i.p. + o \left( \varepsilon^3 \right).
\]

Note that we have also collected all the terms multiplied by \( \left( \tilde{C}_t^* - \tilde{Y}_{F,t} \right) \). Collecting the terms in output gaps and the terms multiplied by output differentials yields:

\[
\mathcal{L}_t^W \times - \left( \frac{\sigma}{2} \tilde{C}_t - \tilde{C}_{C,t} \right) \left( \bar{C}_t - \bar{Y}_{H,t} \right) - \left( \frac{\sigma}{2} \tilde{C}_t^* - \tilde{C}_{C,t}^* + \bar{Q}_t \right) \left( \tilde{C}_t^* - \tilde{Y}_{F,t} \right) -
\]

\[
(1 - a_H) \left( \bar{C}_t - \bar{C}_t^* - \bar{Q}_t \right) \Delta_t -
\]

\[
(1 - a_H) \left[ (\tilde{T}_t + \tilde{\Delta}_t) + \tilde{W}_t + \tilde{\Delta}_t - \tilde{C}_{C,t} - \tilde{C}_{C,t}^* \right] \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) +
\]

\[
(1 - a_H) a_H \phi \left( \tilde{T}_t + \tilde{\Delta}_t \right)^2 + (1 - a_H) \left( [1 - 2a_H (1 - \phi)] \tilde{T}_t - 2a_H (1 - \phi) \tilde{\Delta}_t \right) \tilde{\Delta}_t +
\]

\[
2a_H (1 - a_H) (\sigma \phi - 1) \left( \frac{1}{2} \tilde{T}_t + \tilde{\Delta}_t \right) - \tilde{T}_t^{fb} \right] \tilde{Y}_{H,t} -
\]

\[
(\eta + \sigma) \left( \frac{1}{2} \tilde{Y}_{H,t} - \tilde{Y}_{H,t} \right) \tilde{Y}_{H,t} - (\eta + \sigma) \left( \frac{1}{2} \tilde{Y}_{F,t} - \tilde{Y}_{F,t} \right) \tilde{Y}_{F,t} -
\]

\[
\frac{1}{2} \left( 1 - \alpha \beta \right) \left( 1 - \alpha \right) \left[ a_H \pi_{H,t}^2 + (1 - a_H) \pi_{H,t}^2 + a_H \pi_{F,t}^2 + (1 - a_H) \pi_{F,t}^2 \right] + t.i.p. + o \left( \varepsilon^3 \right).
\]

Using (2) and (3), the first order approximations for \( \bar{C}_t \) and \( \bar{C}_t^* \), we can rearrange
further,
\[
\mathcal{L}_t^W \times
\left[\frac{\sigma}{2} \left( \tilde{C}_t - \tilde{C}^*_t \right) - \tilde{Q}_t - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_{C,t} \right) \right]
\frac{1}{2} \left[ \tilde{Y}_{H,t} - \tilde{Y}_{F,t} - \sigma^{-1} \left[ \tilde{Q}_t + \tilde{W}_t + \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_{C,t} \right) \right] \right] +
(1 - a_h) \left( \tilde{C}_t - \tilde{C}^*_t - \tilde{Q}_t \right) \tilde{\Delta}_t -
(1 - a_h) \frac{1}{2} \left[ \left( \tilde{T}_t + \tilde{\Delta}_t \right) + \left( \tilde{W}_t + \tilde{\Delta}_t \right) - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_{C,t} \right) \right] \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) +
(1 - a_h) a_h \phi \left( \tilde{T}_t + \tilde{\Delta}_t \right)^2 + (1 - a_h) \left( [1 - 2a_h (1 - \phi)] \tilde{T}_t - 2a_h (1 - \phi) \tilde{\Delta}_t \right) \tilde{\Delta}_t +
2a_h (1 - a_h) (\sigma \phi - 1) \frac{1}{2} \left( \tilde{T}_t + \tilde{\Delta}_t \right) - \tilde{\Delta}_t \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) -
(\eta + \sigma) \left( \frac{1}{2} \tilde{Y}_{H,t} - \tilde{Y}_{F,t}^b \right) \tilde{Y}_{H,t} - (\eta + \sigma) \left( \frac{1}{2} \tilde{Y}_{F,t} - \tilde{Y}_{F,t}^b \right) \tilde{Y}_{F,t} -
\frac{1}{2} \frac{\theta_\alpha}{(1 - \alpha \beta) (1 - \alpha)} [a_h \pi_{H,t}^2 + (1 - a_h) \pi_{H,t}^{2*} + a_h \pi_{F,t}^{2*} + (1 - a_h) \pi_{F,t}^2] + t.i.p. + o(\varepsilon^3).
\]

Here is a key passage: using the definition of the demand gap \( \tilde{W}_t = \sigma \left( \tilde{C}_t - \tilde{C}^*_t \right) - \tilde{Q}_t - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_{C,t} \right) \), we can eliminate all the terms in consumption:
\[
\mathcal{L}_t^W \approx \frac{1}{4} \sigma^{-1} \left[ \tilde{W}_t^2 - \left( \tilde{Q}_t + \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_{C,t} \right) \right)^2 \right] +
\frac{1}{4} \left[ \tilde{W}_t - \left( \tilde{Q}_t + \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_{C,t} \right) \right) \right] \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) -
(1 - a_h) \sigma^{-1} \left[ \tilde{Q}_t + \tilde{W}_t + \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_{C,t} \right) \right] \tilde{\Delta}_t -
(1 - a_h) \frac{1}{2} \left[ \left( \tilde{T}_t + \tilde{\Delta}_t \right) + \left( \tilde{D}_t + \tilde{\Delta}_t \right) - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_{C,t} \right) \right] \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) +
(1 - a_h) a_h \phi \left( \tilde{T}_t + \tilde{\Delta}_t \right)^2 + (1 - a_h) [1 - 2a_h (1 - \phi)] \left( \tilde{T}_t + \tilde{\Delta}_t \right) \tilde{\Delta}_t +
2a_h (1 - a_h) (\sigma \phi - 1) \frac{1}{2} \left( \tilde{T}_t + \tilde{\Delta}_t \right) - \tilde{\Delta}_t \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) -
(\eta + \sigma) \left( \frac{1}{2} \tilde{Y}_{H,t} - \tilde{Y}_{F,t}^b \right) \tilde{Y}_{H,t} - (\eta + \sigma) \left( \frac{1}{2} \tilde{Y}_{F,t} - \tilde{Y}_{F,t}^b \right) \tilde{Y}_{F,t} -
\frac{1}{2} \frac{\theta_\alpha}{(1 - \alpha \beta) (1 - \alpha)} [a_h \pi_{H,t}^2 + \pi_{H,t}^{2*} + a_h \pi_{F,t}^{2*} + (1 - a_h) \pi_{F,t}^2] + t.i.p. + o(\varepsilon^3).
\]
We then collect the terms in output differentials:

\[
\mathcal{L}_l^W \times -\frac{1}{4} \sigma^{-1} \left[ \tilde{W}_t^2 - \left( \tilde{Q}_t + \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^{*} \right) \right) \right] -
\]

\[
(1 - a_H) \sigma^{-1} \left( \tilde{Q}_t + \tilde{W}_t + \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^{*} \right) \right) \Delta_t +
\]

\[
\frac{1}{4} \left[ (2a_H - 1) \left( \left( \tilde{W}_t + \Delta_t \right) - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^{*} \right) \right) \right] \left( \tilde{T}_t + \Delta_t \right) \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) +
\]

\[
(1 - a_H) a_H \phi \left( \tilde{T}_t + \Delta_t \right)^2 + (1 - a_H) \left( [1 - 2a_H (1 - \phi)] \tilde{T}_t - 2a_H (1 - \phi) \tilde{\Delta}_t \right) \tilde{\Delta}_t +
\]

\[
2a_H (1 - a_H) (\sigma \phi - 1) \left( \frac{1}{2} \left( \tilde{T}_t + \Delta_t \right) - \tilde{T}_t^{fb} \right) \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) -
\]

\[
(\eta + \sigma) \left( \frac{1}{2} \tilde{Y}_{H,t} - \tilde{Y}_{H,t}^{fb} \right) \tilde{Y}_{H,t} - (\eta + \sigma) \left( \frac{1}{2} \tilde{Y}_{F,t} - \tilde{Y}_{F,t}^{fb} \right) \tilde{Y}_{F,t} -
\]

\[
\frac{1}{2} \frac{\theta \alpha}{(1 - \alpha \beta) (1 - \alpha)} \left[ a_H \pi_H^{2,t} + (1 - a_H) \pi_H^{2,t} + a_H \pi_F^{2,t} + (1 - a_H) \pi_F^{2,t} \right] + t.i.p. + o (\varepsilon^3).
\]

and use the expression for the terms of trade (4) to obtain,

\[
\mathcal{L}_l^W \times -\frac{1}{4} \sigma^{-1} \left[ \tilde{W}_t^2 - \left( \tilde{Q}_t + \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^{*} \right) \right) \right] -
\]

\[
(1 - a_H) \sigma^{-1} \left( \tilde{Q}_t + \tilde{W}_t + \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^{*} \right) \right) \Delta_t +
\]

\[
\frac{1}{4} \left[ (2a_H - 1) \left( \left( \tilde{W}_t + \Delta_t \right) - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^{*} \right) \right) \right] \left( \tilde{T}_t + \Delta_t \right) \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) -
\]

\[
(1 - a_H) a_H \phi \left( \tilde{T}_t + \Delta_t \right)^2 + (1 - a_H) \left( [1 - 2a_H (1 - \phi)] \tilde{T}_t - 2a_H (1 - \phi) \tilde{\Delta}_t \right) \tilde{\Delta}_t -
\]

\[
(\eta + \sigma) \left( \frac{1}{2} \tilde{Y}_{H,t} - \tilde{Y}_{H,t}^{fb} \right) \tilde{Y}_{H,t} - (\eta + \sigma) \left( \frac{1}{2} \tilde{Y}_{F,t} - \tilde{Y}_{F,t}^{fb} \right) \tilde{Y}_{F,t} -
\]

\[
\frac{1}{2} \frac{\theta \alpha}{(1 - \alpha \beta) (1 - \alpha)} \left[ a_H \pi_H^{2,t} + (1 - a_H) \pi_H^{2,t} + a_H \pi_F^{2,t} + (1 - a_H) \pi_F^{2,t} \right] +
\]

\[
2a_H (1 - a_H) (\sigma \phi - 1) \sigma \left( \frac{1}{2} \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) - \left( \tilde{Y}_{H,t}^{fb} - \tilde{Y}_{F,t}^{fb} \right) \right) \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) + t.i.p. + o (\varepsilon^3)
\]
The last three lines of the previous expression coincides with the loss function under complete markets, expressed in deviations from the first best ($\bar{x}_t = \tilde{x}_t - \tilde{x}_t^{fb}$) when also $\Delta_t = 0$—rewritten below for convenience:

\[
\mathcal{L}_t^W - (L_t^W)^{fb} \propto -\frac{1}{2} (\eta + \sigma) \left( \tilde{Y}_{H,t} \right)^2 - \frac{1}{2} (\eta + \sigma) \left( \tilde{Y}_{F,t} \right)^2
\]

\[
-\frac{1}{2} \frac{\theta \alpha}{(1 - \alpha^3)} \left[ a_H \pi_{H,t}^2 + (1 - a_H) \pi_{H,t}^2 + a_H \pi_{F,t}^2 + (1 - a_H) \pi_{F,t}^2 \right] + \frac{a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4 a_H (1 - a_H) (\sigma \phi - 1) + 1} \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right)^2 + t.i.p. + o(\varepsilon^3).
\]

It follows that all the other terms in $\mathcal{L}_t^W$ above must cancel out when $\tilde{W}_t = \tilde{\Delta}_t = 0$. The final step in deriving the generic loss function consists of verifying this conjecture, and derive how our expression must change under incomplete markets and LOOP deviations.

Substitute out for $\tilde{Q}_t$ in terms of $\tilde{T}_t$ and $\tilde{\Delta}_t$ using (1):

\[
-\frac{1}{4} \sigma^{-1} \left[ \tilde{W}_t^2 - \left( (2 a_H - 1) \left( \tilde{T}_t + \tilde{\Delta}_t \right) + \Delta_t + \left( \tilde{C}_{C,t} - \tilde{\zeta}_{C,t} \right) \right)^2 \right] - (1 - a_H) \sigma^{-1} \left[ (2 a_H - 1) \left( \tilde{T}_t + \tilde{\Delta}_t \right) + \left( \Delta_t + \tilde{W}_t \right) + \left( \tilde{C}_{C,t} - \tilde{\zeta}_{C,t} \right) \right] \Delta_t + (1 - a_H) \left( (2 a_H - 1) \left( \tilde{T}_t + \tilde{\Delta}_t \right) + \Delta_t \right) \Delta_t + \frac{1}{4} \left[ (2 a_H - 1) \left( \left( \tilde{W}_t + \tilde{\Delta}_t \right) - \left( \tilde{C}_{C,t} - \tilde{\zeta}_{C,t} \right) \right) - \left( \tilde{T}_t + \tilde{\Delta}_t \right) - 2 \Delta_t \right] \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) - \frac{a_H (1 - a_H) (\sigma \phi - 1)}{4 a_H (1 - a_H) (\sigma \phi - 1) + 1} \left[ (2 a_H - 1) \left( \tilde{W}_t + \tilde{\Delta}_t \right) - \left( \tilde{C}_{C,t} - \tilde{\zeta}_{C,t} \right) \right] \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) + (1 - a_H) a_H \phi \left( \tilde{T}_t + \tilde{\Delta}_t \right)^2 + (1 - a_H) \left( (1 - 2 a_H (1 - \phi)) \tilde{T}_t - 2 a_H (1 - \phi) \tilde{\Delta}_t \right) \tilde{\Delta}_t.
\]
and substitute out the output differential using (4), yielding,
\[
\begin{align*}
&= - \frac{1}{4} \sigma^{-1} \left[ \bar{W}_t^2 - \left( (2a_H - 1) \, (\bar{T}_t + \Delta_t) + \Delta_t + \left( \bar{C}_{C,t} - \bar{\zeta}_{C,t}^* \right)^2 \right) - \\
&\quad (1 - a_H) \, \sigma^{-1} \left[ (2a_H - 1) \, (\bar{T}_t + \Delta_t) + \Delta_t + \bar{W}_t + \left( \bar{C}_{C,t} - \bar{\zeta}_{C,t}^* \right) \right] \Delta_t - \\
&\quad (1 - a_H) \, \sigma^{-1} \sigma \left[ (2a_H - 1) \, (\bar{T}_t + \Delta_t) + \Delta_t \right] \\
&\quad \frac{1}{4} \sigma^{-1} \left[ (2a_H - 1) \left( (\bar{W}_t + \Delta_t) - \left( \bar{C}_{C,t} - \bar{\zeta}_{C,t}^* \right) \right) \right. \\
&\quad \left. - (\bar{T}_t + \Delta_t) - 2\Delta_t \right]
\end{align*}
\]
\[
\begin{align*}
&= \left( \frac{4a_H (1 - a_H) (\sigma \phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right) \left( \bar{W}_t + \Delta_t \right) - \bar{\zeta}_{C,t} + \bar{\zeta}_{C,t}^* \\
&\quad \frac{1}{2} \sigma^{-1} \left[ (4a_H (1 - a_H) (\sigma \phi - 1) + 1) \left( \bar{T}_t + \Delta_t \right) + \bar{W}_t + \left( \bar{C}_{C,t} - \bar{\zeta}_{C,t}^* \right) \right] \Delta_t - \\
&\quad \frac{1}{2} \sigma^{-1} \left[ (2a_H - 1) \left( \bar{T}_t + \Delta_t \right) + \bar{W}_t + \Delta_t + \left( \bar{C}_{C,t} - \bar{\zeta}_{C,t}^* \right) \right] \Delta_t,
\end{align*}
\]
which vanishes under complete markets and PCP. Collecting terms we get,
\[
\begin{align*}
&= - \frac{a_H (1 - a_H) \phi}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \left( \bar{W}_t + \Delta_t \right)^2 - \\
&\quad (1 - a_H) \left( [1 - 2a_H (1 - \phi)] \, \bar{T}_t - 2a_H (1 - \phi) \, \bar{\Delta}_t \right) \bar{\Delta}_t - \\
&\quad (1 - a_H) \sigma^{-1} \left( [1 - \sigma] \, \left( (2a_H - 1) \, (\bar{T}_t + \Delta_t) + \Delta_t \right) + \bar{W}_t + \left( \bar{C}_{C,t} - \bar{\zeta}_{C,t}^* \right) \right) \Delta_t - \\
&\quad \frac{1}{2} \sigma^{-1} \left[ (2a_H - 1 - 4a_H (1 - a_H) (\sigma \phi - 1) + 1) \left( \bar{T}_t + \Delta_t \right) + 2(1 - a_H) \left( 1 - \sigma \right) (2a_H - 1) \left( \bar{T}_t + \Delta_t \right) \right. \\
&\quad \left. + \frac{1}{2} \sigma^{-1} \left[ (2a_H - 1 - 2(1 - a_H) \right] \left( \bar{W}_t + \Delta_t + \left( \bar{C}_{C,t} - \bar{\zeta}_{C,t}^* \right) \right) \right] \Delta_t,
\end{align*}
\]
which further simplifies as follows

\[
= - \frac{\alpha_H (1 - \alpha_H) \phi}{4 \alpha_H (1 - \alpha_H)(\phi - 1) + 1} (\bar{\delta}_t + \Delta_t)^2 + (1 - \alpha_H) \left( [1 - 2 \alpha_H (1 - \phi)] \bar{\delta}_t - 2 \alpha_H (1 - \phi) \bar{\delta}_t \right) \Delta_t + (1 - \alpha_H) [2 \alpha_H (1 - \phi) - 1] \bar{\delta}_t \Delta_t + (1 - \alpha_H) 2 \alpha_H [1 - \phi] \Delta_t^2.
\]

Given that the last three lines cancel out, we conclude that with generically incomplete market under LCP the loss function in deviations from the first best can be expressed as:

\[
\mathcal{L}_t^W - (\mathcal{L}_t^W)^{ib} \propto -\frac{1}{2} (\sigma + \alpha) \left( \bar{\delta}_t \right)^2 - \frac{1}{2} (\eta + \sigma) \left( \bar{\delta}_t \right)^2 + \frac{\theta \alpha}{2 (1 - \alpha^2)} \left[a_H \pi_{H,t}^2 + (1 - a_H) \pi_{H,t}^2 + a_H \pi_{F,t}^2 + (1 - a_H) \pi_{F,t}^2 \right] + \frac{a_H (1 - \alpha_H)}{4 a_H (1 - \alpha_H)(\phi - 1) + 1} \left[ (\sigma \phi - 1) \sigma \left( \bar{\delta}_t \right)^2 - \phi \left( \bar{\delta}_t \right)^2 \right] + t.i.p. + o(\varepsilon^3).
\]

This completes the derivation of the optimal monetary policy loss function in the LCP economy.

### 2.1.3 PCP economy

The loss function under PCP is a special case of the above in which all LOOP deviations \(\Delta_t\) are set to zero, which also implies that the inflation term, \([a_H \pi_{H,t}^2 + (1 - a_H) \pi_{H,t}^2 + a_H \pi_{F,t}^2 + (1 - a_H) \pi_{F,t}^2]\), is equal to \(\pi_{H,t}^2 + \pi_{F,t}^2\).

### 2.1.4 Specifying market incompleteness

Observe that maximization of the world welfare subject to the implementability constraints characterizing the competitive equilibrium requires spelling out the exact form of market incompleteness. Taking the difference of the budget constraints for an economy with \(n\) traded assets we can generically arrive at the following expression:

\[
C_t - Q_t C_t^* = \frac{P_{H,t}^*}{P_t} Y_{H,t} + \left( \frac{S_t P_{F,t}^*}{P_{H,t}} - 1 \right) \frac{P_{H,t}^*}{P_t} C_{H,t}^* - \left( \frac{P_{F,t}^*}{P_t} \right) \left( Q_t Y_{F,t} + \left( 1 - \frac{S_t P_{F,t}^*}{P_{F,t}} \right) \frac{P_{F,t}^*}{P_t} C_{F,t}^* \right) + 2 \left[ (1 + r_t - 1) B_{t-1} + \sum_i \alpha_{i,t-1} (R_{i,t} - (1 + r_t - 1)) - B_t \right],
\]

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Given by:

\[ C_t - Q_t C^*_t = \left[ a_H + (1 - a_H) T_t^{1 - \phi} \Delta_{H,t}^{1 - \phi} \right] \phi Y_{H,t} - \]

\[ \left[ a_H + (1 - a_H) T_{t-1}^{\phi-1} \Delta_{F,t}^{\phi-1} \right] \phi Q_t Y_{F,t} + \]

\[ \left(1 - \frac{P_{H,t}}{S_t P^*_H} \right) \frac{P^*_H S_t P^*_t}{P_t} C_{H,t} + \left( S P^*_t + 1 \right) \frac{P_{F,t}}{P_t} C_{F,t} + \]

\[ 2 \left[ (1 + r_{t-1}) B_{t-1} + \sum_i \alpha_{i,t-1} (R_{i,t} - (1 + r_{t-1})) - B_t \right] \]

and all ex-post returns are expressed in terms of Home consumption prices — e.g. \( 1 + r_{t-1} = \frac{1}{1 + \beta} \frac{P_t}{P_{t-1}} \) and \( \sum_i \alpha_{i,t} = B_t \). Around a symmetric steady state with zero real NFA \((B = 0)\), the consumption differential, up to first order, is given by:

\[ \tilde{C}_t - \tilde{C}^*_t - \tilde{Q}_t = \]

\[ \tilde{Y}_{H,t} - \tilde{Y}_{F,t} - \tilde{Q}_t - 2 (1 - a_H) \tilde{T}_t - (1 - a_H) \left( \tilde{\Delta}_{F,t} + \tilde{\Delta}_{H,t} \right) + \]

\[ (1 - a_H) \left( \tilde{\Delta}_{F,t} + \tilde{\Delta}_{H,t} \right) + 2 \beta^{-1} \left( \tilde{B}_{t-1} - \beta \tilde{B}_t + \sum_i \frac{\omega_i}{Y} \left( \tilde{R}_{i,t} - (1 + r_{t-1}) \right) \right). \]

Where NFA deviations are defined wrt to steady state output \( \tilde{B}_{t-1} = \frac{B_{t-1} - 0}{Y} \), and \( \omega_i \) represents the share of gross wealth invested in the \( i \)-th asset in the stochastic steady state.

For \( \tilde{\Delta}_{H,t} = \tilde{\Delta}_{F,t} = \tilde{\Delta}_t \) under symmetry, we get:

\[ \tilde{C}_t - \tilde{C}^*_t = \tilde{Y}_{H,t} - \tilde{Y}_{F,t} - 2 (1 - a_H) \tilde{T}_t + \]

\[ 2 \beta^{-1} \left( \tilde{B}_{t-1} - \beta \tilde{B}_t + \sum_i \frac{\omega_i}{Y} \left( \tilde{R}_{i,t} - (1 + r_{t-1}) \right) \right). \]

Under financial autarky, since \( \tilde{B}_{t-1} = 0 \), we have the following:

\[ \tilde{\Psi}_t = \sigma \left[ \tilde{Y}_{H,t} - \tilde{Y}_{F,t} - 2 (1 - a_H) \tilde{T}_t \right] - \tilde{Q}_t - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_{C,t} \right). \]
whereas, in the case of a bond economy, the wealth gap $\tilde{W}_t$ will also reflect net capital flows:

$$
\tilde{W}_t = \sigma \left[ -\left( \tilde{B}_t - \beta^{-1} \tilde{B}_{t-1} \right) - \tilde{Y}_{H,t} + (1 - a_H) \tilde{T}_t \right] + 
$$

$$
- \tilde{Q}_t - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^* \right).
$$

Finally, rewriting in terms of gaps (useful when characterizing optimal policy) the wealth gap in a bond economy is given by,

$$
\tilde{W}_t = \sigma \left( C_t - \tilde{C}_t \right) - \tilde{Q}_t
$$

$$
= \sigma \left[ \tilde{Y}_{H,t} - \tilde{Y}_{F,t} + 2 \beta^{-1} \left( \tilde{B}_{t-1} - \beta \tilde{B}_t \right) - 2 a_H \tilde{\Delta}_t - \left[ 2 (1 - a_H) \sigma + (2 a_H - 1) \right] \tilde{T}_t + 
2 (1 - a_H) \left[ (2 a_H (\sigma \phi - 1) + 1 - \sigma) \tilde{T}_t^{fb} - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^* \right) \right],
$$

and under autarky,

$$
\tilde{W}_t = \sigma \left( \tilde{C}_t - \tilde{C}_t^* \right) - \tilde{Q}_t
$$

$$
= \sigma \left[ \tilde{Y}_{H,t} - \tilde{Y}_{F,t} - 2 a_H \tilde{\Delta}_t - \left[ 2 (1 - a_H) \sigma + (2 a_H - 1) \right] \tilde{T}_t + 
2 (1 - a_H) \left[ (2 a_H (\sigma \phi - 1) + 1 - \sigma) \tilde{T}_t^{fb} - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^* \right) \right].
$$
2.2 Characterizing optimal monetary targeting rules under incomplete markets

In this section we work out the constrained efficient allocation in our model economy—this is found by maximizing the expected discounted value of the following loss function in deviation from first best,

\[
L^W_t = (L^W_t)^{fb} - \frac{1}{2} (\eta + \sigma) \left( \hat{Y}_{H,t} - \hat{Y}_{H,t}^{fb} \right)^2 - \frac{1}{2} (\eta + \sigma) \left( \hat{Y}_{F,t} - \hat{Y}_{F,t}^{fb} \right)^2 - (8)
\]

\[
\frac{1}{2} \frac{\theta \alpha}{(1 - \alpha \beta)(1 - \alpha)} \left[ a_H \pi_{H,t}^2 + (1 - a_H) \pi_{H,t}^{*2} + a_H \pi_{F,t}^{*2} + (1 - a_H) \pi_{F,t}^{*2} \right] + a_H (1 - a_H) (\sigma \phi - 1) \frac{1}{2} \left[ \left( \hat{Y}_{H,t} - \hat{Y}_{H,t}^{fb} \right) - \left( \hat{Y}_{F,t} - \hat{Y}_{F,t}^{fb} \right) \right]^2 -
\]

\[
a_H (1 - a_H) \phi \frac{1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \left( \hat{\Delta}_t + \hat{W}_t \right)^2 + t.i.p. + o (\varepsilon^3),
\]

with respect to its arguments \( \hat{Y}_{H,t}, \hat{Y}_{F,t}, \hat{D}_t \), and \( \pi_{H,t}, \pi_{F,t}^{*} \), subject to the NK Phillips curve, the equilibrium condition linking relative prices to output gap differentials and demand gaps, the definition of the wealth gap, and the Euler equation characterizing the evolution of the wealth gap. In the case of non-trivial portfolio decisions, higher order Euler equations characterizing these choices must also be considered.

We treat the cases of PCP and LCP separately as some of the constraints differ significantly.

2.2.1 PCP

The PCP loss function is given by (1) subject to (\( \hat{\Delta}_t = 0 \) and \( a_H \pi_{H,t}^2 + (1 - a_H) \pi_{H,t}^{*2} + a_H \pi_{F,t}^{*2} + (1 - a_H) \pi_{F,t}^{*2} = \pi_{H,t}^2 + \pi_{F,t}^{*2} \)). Under PCP optimal monetary policy minimizes the loss function subject to:

1. NK Phillips curves determining inflation rates

\[
\pi_{H,t} = \beta E_t \pi_{H,t+1} + \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha} \left\{ (\eta + \sigma) \left( \hat{Y}_{H,t} \right) - \hat{Y}_{H,t}^{fb} \right\} + \mu_t +
\]

\[
\left\{ - (1 - a_H) \left[ 2a_H (\sigma \phi - 1) \left( \hat{\mu}_t - \hat{\mu}_t^{fb} \right) - \hat{W}_t \right] \right\}
\]

\[
\pi_{F,t}^* = \beta E_t \pi_{F,t+1}^* + \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha} \left\{ (\eta + \sigma) \left( \hat{Y}_{F,t} \right) - \hat{Y}_{F,t}^{fb} \right\} + \mu_t^* +
\]

\[
\left\{ + (1 - a_H) \left[ 2a_H (\sigma \phi - 1) \left( \hat{\mu}_t - \hat{\mu}_t^{fb} \right) - \hat{W}_t \right] \right\},
\]

where the equilibrium relations for first best outcomes \( \hat{Y}_{H,t}^{fb}, \hat{Y}_{F,t}^{fb}, \hat{\mu}_t^{fb} \) in terms
of fundamental shocks are as follows:

\[
(\eta + \sigma) \tilde{Y}^{fb}_{H,t} = [2a_H (1 - a_H) (\sigma \phi - 1)] (\tilde{T}^{fb}_t) - (1 - a_H) \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_C \right) + \tilde{\zeta}_{C,t} + (1 + \eta) \tilde{\zeta}_{Y,t},
\]

\[
(\eta + \sigma) \tilde{Y}^{fb}_{F,t} = [2a_H (1 - a_H) (\sigma \phi - 1)] \left( -\tilde{T}^{fb}_t \right) + (1 - a_H) \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_C \right) + \tilde{\zeta}^*_C + (1 + \eta) \tilde{\zeta}^*_Y,
\]

whereas the terms of trade can in turn be written as a function of relative output and preference shocks

\[
[4 (1 - a_H) \sigma + (2a_H - 1)^2] \tilde{T}^{fb}_t = \sigma \left( \tilde{Y}^{fb}_{H,t} - \tilde{Y}^{fb}_{F,t} \right) - (2a_H - 1) \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_C
\]

2. The equilibrium condition linking relative prices to output differentials and the wealth gap:

\[
\tilde{T} - \tilde{T}^{fb}_t = \frac{\sigma \left[ (\tilde{Y}^{fb}_{H,t} - \tilde{Y}^{fb}_{F,t}) - (2a_H - 1) \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_C \right]}{4a_H (1 - a_H)(\sigma \phi - 1) + 1};
\]

3. The definition of demand gap \( \tilde{W}_t \) in terms of differences in budget constraints and real net wealth \( \tilde{B}_t \):

\[
\tilde{W}_t = \tilde{W}_t = \sigma \left( \tilde{C}_t - \tilde{C}^*_t \right) - \tilde{Q}_t - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_C \right)
\]

\[
= \sigma \left[ \tilde{Y}^{fb}_{H,t} - \tilde{Y}^{fb}_{F,t} - 2 (1 - a_H) \tilde{T}^{fb}_t \right] + \frac{2 \beta^{-1} \left( \tilde{B}_{t-1} - \tilde{B}_t \right)}{\beta - \tilde{B}_t} + (2a_H - 1) \tilde{T} - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_C \right);
\]

4. the Euler equations characterizing the evolution of \( \tilde{W}_t \) (and net wealth \( \tilde{B}_t \)):

\[
E_t \tilde{W}_{t+1} = \tilde{W}_t.
\]

**Bond economy**

Observe that in the case of a bond economy, the program amounts to choosing \( \tilde{Y}^{fb}_{H,t}, \tilde{Y}^{fb}_{F,t}, \tilde{D}_t, \pi^{fb}_{H,t}, \pi^{fb}_{F,t} \) and \( \tilde{B}_t \) subject to the following expression for \( \tilde{W}_t \) in terms of differences of budget constraints:

\[
(1 - a_H) [1 + 2a_H (\phi - 1)] \tilde{W}_t = [4a_H (1 - a_H) (\sigma \phi - 1) + 1] \left( \beta^{-1} \tilde{B}_{t-1} - \tilde{B}_t \right) + (1 - a_H) [2a_H (\sigma \phi - 1) + 1 - \sigma] \left[ (\tilde{Y}^{fb}_{H,t} - \tilde{Y}^{fb}_{H,t}) - (\tilde{Y}^{fb}_{F,t} - \tilde{Y}^{fb}_{F,t}) \right] + (1 - a_H) [4a_H (1 - a_H) (\sigma \phi - 1) + 1] \sigma^{-1} \left[ (2a_H (\sigma \phi - 1) + 1 - \sigma) \tilde{T}^{fb}_t - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}^*_C \right) \right];
\]
The necessary FOC’s with respect to inflation are given by:

\[
\begin{align*}
\pi_{H,t} : \quad 0 &= -\frac{\alpha}{(1-\alpha\beta)(1-\alpha)} \pi_{H,t} - \gamma_{H,t} + \gamma_{H,t-1} \\
\pi_{F,t} : \quad 0 &= -\frac{\alpha}{(1-\alpha\beta)(1-\alpha)} \pi_{F,t} - \gamma_{F,t} + \gamma_{F,t-1},
\end{align*}
\]

implying

\[
\begin{align*}
-\frac{(1-\alpha\beta)(1-\alpha)}{\alpha} (\gamma_{H,t} - \gamma_{H,t-1}) &= \theta \pi_{H,t} = \theta (\tilde{p}_{H,t} - \tilde{p}_{H,t-1}) \\
-\frac{(1-\alpha\beta)(1-\alpha)}{\alpha} (\gamma_{F,t} - \gamma_{F,t-1}) &= \theta (\tilde{p}_{F,t} - \tilde{p}_{F,t-1}),
\end{align*}
\]

where \(\gamma_{H,t}\) and \(\gamma_{F,t}\) are the multipliers associated with the Phillips curves — whose lags appear reflecting the assumption of commitment; and with respect to output (where observe that we have switched to the gap notation, e.g. \(\tilde{Y}_{H,t} = \tilde{Y}_{H,t} - \tilde{Y}_{H,t}^{f}\)):

\[
\begin{align*}
\tilde{Y}_{H,t} : \quad 0 &= (\sigma + \eta) \tilde{Y}_{H,t} - \\
&\quad \frac{2a_{H}(1-a_{H})(\sigma\phi - 1)\sigma}{4a_{H}(1-a_{H})(\sigma\phi - 1) + 1} \left[ \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right] + \\
&\quad \frac{2a_{H}(1-a_{H})}{4a_{H}(1-a_{H})(\sigma\phi - 1) + 1} \tilde{W}_{t} \\
&\quad \frac{2a_{H}(\sigma\phi - 1) + 1 - \sigma}{2a_{H}(\phi - 1) + 1} \left( \lambda_{t} - \beta^{-1}\lambda_{t-1} \right) - \\
&\quad \frac{\sigma + \eta}{2a_{H}(\sigma - 1) + 1} (1-\alpha\beta)(1-\alpha) \frac{\alpha}{\gamma_{H,t}} - \\
&\quad (1-a_{H})(\sigma - 1)(1-\alpha\beta)(1-\alpha) \frac{\alpha}{\gamma_{F,t}};
\end{align*}
\]

\[
\begin{align*}
\tilde{Y}_{F,t} : \quad 0 &= (\sigma + \eta) \tilde{Y}_{F,t} + \\
&\quad \frac{2a_{H}(1-a_{H})(\sigma\phi - 1)\sigma}{4a_{H}(1-a_{H})(\sigma\phi - 1) + 1} \left[ \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right] - \\
&\quad \frac{2a_{H}(1-a_{H})}{4a_{H}(1-a_{H})(\sigma\phi - 1) + 1} \tilde{W}_{t} - \\
&\quad \frac{2a_{H}(\sigma\phi - 1) + 1 - \sigma}{2a_{H}(\phi - 1) + 1} \left( \lambda_{t} - \beta^{-1}\lambda_{t-1} \right) - \\
&\quad \frac{\sigma + \eta}{2a_{H}(\phi - 1) + 1} (1-\alpha\beta)(1-\alpha) \frac{\alpha}{\gamma_{F,t}} - \\
&\quad (1-a_{H})(\sigma - 1)(1-\alpha\beta)(1-\alpha) \frac{\alpha}{\gamma_{H,t}};
\end{align*}
\]
Furthermore,

\[ \tilde{B}_t : 0 = 2a_H (1 - a_H) \phi \left[ E_t \tilde{W}_{t+1} - \tilde{W}_t \right] + \\
\left[ 4a_H (1 - a_H) (\sigma \phi - 1) + 1 \right] \left[ (E_t \lambda_{t+1} - \lambda_t) - \beta^{-1} (\lambda_t - \lambda_{t-1}) \right] - \\
(1 - a_H) \left[ 2a_H (\sigma \phi - 1) + 1 \right] \left( \frac{1 - \alpha \beta}{\alpha} \right) . \\
\left[ (E_t \gamma_{H,t+1} - \gamma_{H,t}) \left( E_t \gamma_{F,t+1} - \gamma_{F,t} \right) \right] \]

implying

\[ 0 = [(\beta E_t \lambda_{t+1} - \lambda_t) - (\beta \lambda_t - \lambda_{t-1})] + \\
(1 - a_H) \left( \frac{2a_H (\sigma \phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right) \theta \left( \beta E_t \pi_{H,t+1} - \beta E_t \pi_{F,t+1} \right) . \]

The solution can be expressed in terms of a familiar sum rule for (the change in) world output gaps and inflation rates:

\[ 0 = \tilde{Y}_{H,t} + \tilde{Y}_{F,t} + \theta \left( \tilde{p}_{H,t} + \tilde{p}_{F,t} \right) \\
= \left[ \tilde{Y}_{H,t} - \tilde{Y}_{H,t-1} \right] + \left[ \tilde{Y}_{F,t} - \tilde{Y}_{F,t-1} \right] + \\
\theta \left[ \pi_{H,t} + \pi_{F,t} \right] , \]

and a difference rule which can be obtained by subtracting the output FOC’s to solve for \( \lambda_t \):

\[ -2a_H (\sigma \phi - 1) + 1 - \sigma \beta^{-1} (\beta \lambda_t - \lambda_{t-1}) = \\
\left[ (\sigma + \eta) - \frac{4a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) + \\
\frac{4a_H (1 - a_H) \phi}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \left( \tilde{y}_{H,t} - \tilde{y}_{F,t} \right) + \\
\left[ \frac{4a_H (1 - a_H) \phi}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \theta \left( \tilde{p}_{H,t} - \tilde{p}_{F,t} \right) . \]

We can solve for \( (\beta \lambda_t - \lambda_{t-1}) \) from the first order condition for \( \tilde{B}_t \)

\[ 0 = [(\beta E_t \lambda_{t+1} - \lambda_t) - (\beta \lambda_t - \lambda_{t-1})] + \\
(1 - a_H) \left( \frac{2a_H (\sigma \phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right) \theta \beta \left( E_t \pi_{H,t+1} - E_t \pi_{F,t+1} \right) , \]

\[- \left[ E_t (\beta \lambda_{t+1} - \lambda_t) - (\beta \lambda_t - \lambda_{t-1}) \right] = \\
(1 - a_H) \left( \frac{2a_H (\sigma \phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right) \theta \beta E_t \left[ \left( \tilde{p}_{H,t+1} - \tilde{p}_{H,t} \right) - \left( \tilde{p}_{F,t+1} - \tilde{p}_{F,t} \right) \right] . \]
A solution to the above equation is given by the following:

\[-(\beta \lambda_t - \lambda_{t-1}) = (1 - a_H) \left( \frac{2a_H (\sigma \phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right) \theta \beta \left( \tilde{p}_{H,t} - \tilde{p}_{F,t} \right).\]

Effectively this assumes that the growth rate in the (quasi-change \((\beta \lambda_t - \lambda_{t-1})\) of the) Lagrange multiplier of relative wealth depends on contemporaneous shocks only via their effects on inflation differentials.

In turn, this implies the following difference rule:

\[0 = \left[ (\sigma + \eta) - \frac{4a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) + \frac{4a_H (1 - a_H) \phi}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \tilde{W}_t + \frac{\sigma + \eta - \frac{4a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1}}{2a_H (\phi - 1) + 1} \theta \left( \tilde{p}_{H,t} - \tilde{p}_{F,t} \right).\]

Therefore, in terms of inflation rates and growth rates the "difference" rule is the following:

\[0 = \left[ (\sigma + \eta) - \frac{4a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left\{ \frac{\left( \tilde{Y}_{H,t} - \tilde{Y}_{H,t-1} \right) - \left( \tilde{Y}_{F,t} - \tilde{Y}_{F,t-1} \right)}{\theta \left( \pi_{H,t} - \pi_{F,t} \right)} \right\} + \frac{4a_H (1 - a_H) \phi}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \left( \tilde{W}_t - \tilde{W}_{t-1} \right).\]

Alternatively, we can substitute the above expression for the Lagrange multiplier \(\lambda_t\)

\[-2a_H (\sigma \phi - 1) + 1 - \frac{\sigma}{2a_H (\phi - 1) + 1} \beta^{-1} (\beta \lambda_t - \lambda_{t-1}) = \]

\[\left[ (\sigma + \eta) - \frac{4a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left( \tilde{Y}_{H,t} - \tilde{Y}_{F,t} \right) + \frac{4a_H (1 - a_H) \phi}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \tilde{W}_t + \left[ \sigma + \eta - \frac{2(1 - a_H) (\sigma - 1)}{2a_H (\phi - 1) + 1} \right] \theta \left( \tilde{p}_{H,t} - \tilde{p}_{F,t} \right),\]

into the first order condition for \(\tilde{B}_t\),

\[0 = \left[ (\beta E_t \lambda_{t+1} - \lambda_t) - (\beta \lambda_t - \lambda_{t-1}) \right] + (1 - a_H) \left( \frac{2a_H (\sigma \phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right) \theta \left( \beta E_t \pi_{H,t+1} - \beta E_t \pi_{F,t+1} \right).\]
yielding:

\[ 0 = -2 (1 - a_H) \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \cdot \\
\left( \frac{2a_H (\sigma \phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right) \theta \left( E_t \pi_{H,t+1} - E_t \pi^*_F,t+1 \right) + \\
\left[ (\sigma + \eta) - \frac{4a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left[ E_t \left( \tilde{Y}_{H,t+1} - \tilde{Y}_{H,t} \right) + \\
- E_t \left( \tilde{Y}_{F,t+1} - \tilde{Y}_{F,t} \right) \right] + \\
\frac{4a_H (1 - a_H) \phi}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \left( E_t \tilde{W}_{t+1} - \tilde{W}_t \right) + \\
\left[ \sigma + \eta - 2 \frac{(1 - a_H) (\sigma - 1)}{2a_H (\phi - 1) + 1} \right] \theta \left( E_t \pi_{H,t+1} - E_t \pi^*_F,t+1 \right). \]

Recalling the law of motion for the wealth gap, \( E_t \tilde{W}_{t+1} = \tilde{W}_t \) and the expression,

\[ \frac{2 (1 - a_H) (\sigma - 1)}{2a_H (\phi - 1) + 1} = \frac{4a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1}, \]

we obtain the following targeting rule:

\[ 0 = \left[ \sigma + \eta - \frac{4a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left[ E_t \left( \tilde{Y}_{H,t+1} - \tilde{Y}_{H,t} \right) + \\
- E_t \left( \tilde{Y}_{F,t+1} - \tilde{Y}_{F,t} \right) \right] + \\
\left\{ E_t \left[ \left( \tilde{Y}_{H,t+1} - \tilde{Y}_{H,t} \right) - \left( \tilde{Y}_{F,t+1} - \tilde{Y}_{F,t} \right) \right] + \\
\theta \left( \pi_{H,t+1} - \pi^*_F,t+1 \right) \right\}. \]

Interestingly, this rule is a forward-looking version of the one which prevails under complete markets:

\[ 0 = \left[ \tilde{Y}_{H,t} - \tilde{Y}_{H,t-1} \right] + \left[ \tilde{Y}_{F,t} - \tilde{Y}_{F,t-1} \right] + \\
\theta \left( \pi_{H,t} - \pi^*_F,t \right). \]

**Solving explicitly for the allocation under the optimal policy**  
The targeting rule can thus be written:

\[ 0 = \left[ \sigma + \eta - \frac{4a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left[ \tilde{Y}_{H,t} - \tilde{Y}_{H,t-1} \right] + \\
\left( \tilde{W}_t - \tilde{W}_{t-1} \right) \left( \sigma + \eta - \frac{4a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right) \left[ \tilde{Y}_{H,t} - \tilde{Y}_{H,t-1} \right] + \\
\theta \left( \pi_{H,t} - \pi^*_F,t \right). \]
Using it to solve for inflation and substituting into the Phillips curve:

\[
\theta \pi_{H,t} = - \left( \bar{Y}_{H,t} - \bar{Y}_{H,t-1} \right) + \\
\frac{2a_H (\phi - 1)}{\eta (4a_H (1 - a_H) (\sigma \phi - 1) + 1) + \sigma} \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \left( \bar{W}_t - \bar{W}_{t-1} \right),
\]
and recalling the following relation for \( \bar{W}_t \):

\[
(1 - a_H) \left[ \frac{2a_H (\phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \bar{W}_t = -\beta^{-1} \left( \beta \hat{B}_t - \hat{B}_{t-1} \right) + \\
(1 - a_H) \left[ \frac{2a_H (\sigma \phi - 1) - (\sigma - 1)}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left[ 2\bar{Y}_{H,t} + \left( \bar{f}^{fb}_{H,t} - \bar{f}^{fb}_{F,t} \right) \right] - \\
(1 - a_H) \left[ \frac{2a_H (\phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left( \zeta_{C,t} - \zeta^{*}_{C,t} \right)
\]

we obtain the following system of difference equations in \( \bar{Y}_{H,t} \) and \( \bar{B}_t \):

\[
\beta^{-1} \left[ E_t \left( \beta \hat{B}_{t+1} - \hat{B}_t \right) - \left( \beta \hat{B}_t - \hat{B}_{t-1} \right) \right] - \\
2 (1 - a_H) \left[ \frac{2a_H (\sigma \phi - 1) - (\sigma - 1)}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left[ E_t \left( \bar{Y}_{H,t+1} - \bar{Y}_{H,t} \right) \right] = \\
(1 - a_H) \left[ \frac{2a_H (\sigma \phi - 1) - (\sigma - 1)}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left[ E_t \left( \bar{f}^{fb}_{H,t+1} - \bar{f}^{fb}_{F,t+1} \right) - \left( \bar{f}^{fb}_{H,t} - \bar{f}^{fb}_{F,t} \right) \right] - \\
(1 - a_H) \left[ \frac{2a_H (\phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left[ E_t \left( \zeta_{C,t+1} - \zeta^{*}_{C,t+1} \right) - \left( \zeta_{C,t} - \zeta^{*}_{C,t} \right) \right],
\]

and,

\[
\left\{ \begin{array}{l}
- \left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left[ \bar{Y}_{H,t} - \bar{Y}_{H,t-1} \right] - \\
\frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \left( \bar{W}_t - \bar{W}_{t-1} \right)
\end{array} \right\}
\]

\[
+ \beta \left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] E_t \left[ \bar{Y}_{H,t+1} - \bar{Y}_{H,t} \right] = \\
\frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] E_t \left[ \bar{Y}_{H,t+1} - \bar{Y}_{H,t} \right] + \\
\frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \bar{W}_t.
\]

We use the method of undetermined coefficients to solve this system, exploiting the martingale nature of the variable \( \bar{W}_t \), namely \( E_t \bar{W}_{t+j} = \bar{W}_t \).
Rearranging the last difference equation for the output gap as follows:

\[
\beta E_t \left[ \bar{Y}_{H,t+1} - \bar{Y}_{H,t} \right] - \left[ \bar{Y}_{H,t} - \bar{Y}_{H,t-1} \right] - 
\]

\[
\left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \delta \bar{Y}_{H,t}
\]

\[
= \frac{2a_H (1 - a_H) \phi}{\eta [4a_H (1 - a_H) (\sigma \phi - 1) + 1] + \sigma} \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \left( \bar{W}_t - \bar{W}_{t-1} \right) + (1 - \alpha) \frac{2a_H (\sigma \phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \bar{W}_t,
\]

we can solve for \(Y_{H,t}\) as function of current and future values of \(W_t\):

\[
\bar{Y}_{H,t} - \delta_1 \bar{Y}_{H,t-1} = 
\]

\[
- \left[ \frac{(1 - \alpha) \eta [4a_H (1 - a_H) (\sigma \phi - 1) + 1] + \sigma}{\sum_{j=0}^{\infty} \delta_2^j E_t (\bar{W}_{t+j} - \bar{W}_{t+j-1})} \right] 
\]

\[- (1 - \alpha) \frac{2a_H (\sigma \phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \theta \sum_{j=0}^{\infty} \delta_2^j E_t \bar{W}_{t+j},\]

where \(0 < \delta_1 < 1 < \beta^{-1} < \delta_2\) are the eigenvalues of the difference equation, solving the standard characteristic equation:

\[
\beta \delta^2 - \left\{ 1 + \beta + \left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \right\} \delta + 1 = 0,
\]

namely,

\[
\delta = \frac{1}{2\beta} \left\{ 1 + \beta + \left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \right\} \pm \frac{1}{2\beta} \sqrt{\left\{ 1 + \beta + \left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \right\}^2 - 4\beta},
\]

We can simplify the above solution which is solely a function of \(W_t\), as \(E_t \bar{W}_{t+j} = \bar{W}_t\):

\[
\left( \bar{Y}_{H,t+j} - \bar{Y}_{H,t+j-1} \right) = 
\]

\[
\delta_1 \left( \bar{Y}_{H,t+j-1} - \bar{Y}_{H,t+j-1} \right) - 
\]

\[
\frac{2a_H \phi}{\eta [4a_H (1 - a_H) (\sigma \phi - 1) + 1] + \sigma} \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \frac{1}{\beta \delta_2} E_t \left( \bar{W}_{t+j} - \bar{W}_{t+j-1} \right) - 
\]

\[
(1 - \alpha) \frac{2a_H (\sigma \phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \theta \frac{1}{\beta (\delta_2 - 1)} \bar{W}_t;
\]
and we have that
\[
\frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \left[ \eta + \frac{\sigma}{4 \alpha H (1 - \alpha H) (\sigma \phi - 1) + 1} \right] = \frac{(\delta_2 - 1) (\beta \delta_2 - 1)}{\delta_2}
\]

Furthermore,
\[
E_t \hat{Y}_{H,t+s} = \\
\delta_1 \left[ (\hat{Y}_{H,t+s-1} - \hat{Y}_{H,t+s-1}^{fb}) \right] - (1 - \alpha H) \frac{2 \alpha H (\sigma \phi - 1) + 1}{\eta [4 \alpha H (1 - \alpha H) (\sigma \phi - 1) + 1] + \sigma} \hat{W}_t + \\
- (1 - \alpha H) \frac{2 \alpha H (\sigma \phi - 1) + 1 - \sigma}{\eta [4 \alpha H (1 - \alpha H) (\sigma \phi - 1) + 1] + \sigma} \frac{1}{\beta \delta_2} E_t \left( \hat{W}_{t+s} - \hat{W}_{t+s-1} \right)
\]

Notice that the second term \( E_t (\hat{W}_{t+j} - \hat{W}_{t+j-1}) \) is 0 for \( j \geq 1 \), while it is equal to \( \hat{W}_t \) for \( j = 0 \). The last term represents a constant shifter proportional to \( \hat{W}_t \) for any \( j \geq 0 \).

We can compare the above with the allocation under \( \pi_{H,t} = \pi_{F,t} = 0 \), characterized as follows:
\[
\left( \hat{\pi}_{H} - \hat{\pi}_t^{fb} \right) = - \frac{\sigma + (2 \alpha H - 1) \eta}{\eta [4 \alpha H (1 - \alpha H) (\sigma \phi - 1) + 1] + \sigma} \hat{W}_t \\
\left( \hat{Y}_{H,t} - \hat{Y}_{H,t}^{fb} \right) = - (1 - \alpha H) \frac{1 + 2 \alpha H (\sigma \phi - 1)}{\eta [4 \alpha H (1 - \alpha H) (\sigma \phi - 1) + 1] + \sigma} \hat{W}_t.
\]

We can also solve for inflation using the targeting rule:
\[
\theta \pi_{H,t} = - \left[ (\hat{\pi}_{H,t} - \hat{\pi}_t^{fb}) - (\hat{Y}_{H,t-1} - \hat{Y}_{H,t-1}^{fb}) \right] + \\
- \frac{2 \alpha H (1 - \alpha H) \phi}{\eta [4 \alpha H (1 - \alpha H) (\sigma \phi - 1) + 1] + \sigma} \frac{2 \alpha H (\sigma \phi - 1) + 1 - \sigma}{2 \alpha H (\phi - 1) + 1} \left( \hat{W}_t - \hat{W}_{t-1} \right),
\]

which implies:
\[
\theta E_t \pi_{H,t} + j = (1 - \delta_1) \left( \hat{Y}_{H,t+j-1} - \hat{Y}_{H,t+j-1}^{fb} \right) + \\
(1 - \alpha H) \frac{2 \alpha H (\sigma \phi - 1) + 1}{\eta [4 \alpha H (1 - \alpha H) (\sigma \phi - 1) + 1] + \sigma} \frac{\theta}{\beta (\delta_2 - 1)} \hat{W}_t + \\
- \frac{2 \alpha H (1 - \alpha H) \phi}{\eta [4 \alpha H (1 - \alpha H) (\sigma \phi - 1) + 1] + \sigma} \frac{2 \alpha H (\sigma \phi - 1) + 1 - \sigma}{2 \alpha H (\phi - 1) + 1} \frac{(\beta \delta_2 - 1)}{\beta \delta_2} E_t \left( \hat{W}_{t+j} - \hat{W}_{t+j-1} \right).
\]

Likewise, armed with the above solution for \( \left( \hat{Y}_{H,t+j} - \hat{Y}_{H,t+j}^{fb} \right) \), we can solve
the following difference equation for $\hat{B}_t$:

$$\beta E_t \left( \hat{B}_{t+1} - \hat{B}_t \right) - \left( \hat{B}_t - \hat{B}_{t-1} \right) =$$

$$2 (1 - a_H) \left[ \frac{2a_H (\sigma \phi - 1) - (\sigma - 1)}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \beta \left[ E_t \hat{Y}_{H,t+1} - \hat{Y}_{H,t} \right] +$$

$$(1 - a_H) \left[ \frac{2a_H (\sigma \phi - 1) - (\sigma - 1)}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \beta \left[ E_t \left( \hat{\gamma}_{H,t+1}^b - \hat{\gamma}_{F,t+1}^b \right) \right] +$$

$$(1 - a_H) \left[ \frac{2a_H (\phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \beta \left[ E_t \left( \hat{\zeta}_{C,t+1} - \hat{\zeta}_{C,t} \right) \right].$$

The eigenvalues of this difference equation are 1 and $1/\beta$, yielding the following standard solution:

$$\hat{B}_t = \left[ \sum_{j=0}^{\infty} \beta^j E_t \left( \hat{Y}_{H,t+j+1} - \hat{Y}_{H,t+j} \right) \right] -$$

$$(1 - a_H) \left[ \frac{2a_H (\sigma \phi - 1) - (\sigma - 1)}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \beta \sum_{j=0}^{\infty} \beta^j E_t \left( \hat{Y}_{H,t+j+1} - \hat{Y}_{H,t+j} \right) -$$

$$(1 - a_H) \left[ \frac{2a_H (\phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \beta \sum_{j=0}^{\infty} \beta^j E_t \left( \hat{\zeta}_{C,t+j+1} - \hat{\zeta}_{C,t+j} \right).$$

Using the above solution for the output gap, we have that for $j \geq 0$:

$$E_t \left( \hat{Y}_{H,t+j+1} - \hat{Y}_{H,t+j} \right) =$$

$$- (1 - \delta_1) E_t \left( \hat{Y}_{H,t+j+1} - \hat{Y}_{H,t+j} \right) +$$

$$-(1 - a_H) \left[ \frac{2a_H (\sigma \phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \left( 1 - \alpha \beta \right) \left( 1 - \alpha \right) \frac{1}{\beta (\delta_2 - 1)} \hat{W}_t,$$

where

$$E_t \left( \hat{Y}_{H,t+j+1} - \hat{Y}_{H,t+j} \right) =$$

$$\delta_1 \left[ \hat{Y}_{H,t+1} - \hat{Y}_{H,t-1} \right] - \frac{2a_H (1 - a_H) \phi}{\eta \left[ 4a_H (1 - a_H) (\sigma \phi - 1) + 1 \right] + \sigma} \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \frac{1}{\beta \delta_2} \hat{W}_t -$$

$$(1 - a_H) \frac{2a_H (\sigma \phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \left( 1 - \alpha \beta \right) \left( 1 - \alpha \right) \frac{1}{\beta (\delta_2 - 1)} \sum_{s=0}^{j} \delta_1^s \hat{W}_t.$$
which also implies that:

\[
E_t \left[ \left( \bar{Y}_{H,t+j+1} - \bar{Y}^{fb}_{H,t+j+1} \right) - \left( \bar{Y}_{H,t+j} - \bar{Y}^{fb}_{H,t+j} \right) \right] = \\
- \left( 1 - \delta_1 \right) \delta_1 \left\{ \delta_1 \left( \bar{Y}_{H,t-1} - \bar{Y}^{fb}_{H,t-1} \right) \right\} \\
- \left( 1 - \delta_1 \right) \delta_1 \left\{ \frac{2a_H (1 - a_H) \phi}{\eta} \right\} \\
- \left( 1 - \delta_1 \right) \delta_1 \left\{ \frac{2a_H (1 - a_H) (\sigma - 1) + 1}{\eta} \right\} \\
\left( 1 - a_H \right) \left\{ \frac{2a_H (\sigma - 1) + 1}{\eta} \right\} \left( 1 - a_H \right) \left( (\alpha - \beta) \right) \left( 1 - (1 - \delta_1) \frac{1 - \beta \delta_2}{1 - \delta_1} \right) \left[ \bar{W}_t \right].
\]

Therefore the solution for NFA is the following:

\[
\hat{B}_t = \hat{B}_{t-1} + \left( 1 - a_H \right) \left\{ \frac{2a_H (\sigma - 1) + 1}{\eta} \right\} \left( 1 - a_H \right) \left( (\alpha - \beta) \right) \left( 1 - (1 - \delta_1) \frac{1 - \beta \delta_2}{1 - \delta_1} \right) \left[ \bar{W}_t \right].
\]

As a result we have that:

\[
\beta \sum_{j=0}^{\infty} \beta^j E_t \left[ \left( \bar{Y}_{H,t+j+1} - \bar{Y}^{fb}_{H,t+j+1} \right) - \left( \bar{Y}_{H,t+j} - \bar{Y}^{fb}_{H,t+j} \right) \right] = \\
- \left( 1 - \delta_1 \right) \beta \sum_{j=0}^{\infty} \beta^j \delta_1 \left\{ \delta_1 \left( \bar{Y}_{H,t-1} - \bar{Y}^{fb}_{H,t-1} \right) \right\} \\
- \left( 1 - \delta_1 \right) \delta_1 \left\{ \frac{2a_H (1 - a_H) \phi}{\eta} \right\} \\
- \left( 1 - \delta_1 \right) \delta_1 \left\{ \frac{2a_H (1 - a_H) (\sigma - 1) + 1}{\eta} \right\} \\
\left( 1 - a_H \right) \left\{ \frac{2a_H (\sigma - 1) + 1}{\eta} \right\} \left( 1 - a_H \right) \left( (\alpha - \beta) \right) \left( 1 - (1 - \delta_1) \frac{1 - \beta \delta_2}{1 - \delta_1} \right) \left[ \bar{W}_t \right].
\]
Finally, recalling the following relation for $\hat{W}_t$:

$$(1 - a_H) \left[ \frac{2a_H (\phi - 1) + 1}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \hat{W}_t = -\beta^{-1} \left( \beta \hat{B}_t - \hat{B}_{t-1} \right) +$$

we can solve for the impact response on $\hat{W}_t$ for $j = 0$ as a function only of exogenous shocks. The permanent response of the wealth gap under the optimal policy is given by:

$$\hat{W}_t \left[ 2a_H (\phi - 1) + 1 \right] + \frac{2(1 - a_H) [2a_H (\sigma \phi - 1) - (\sigma - 1)]}{\eta [4a_H (1 - a_H) (\sigma \phi - 1) + 1] + \sigma} \frac{1}{\beta} \left[ \left( \frac{2a_H (\sigma \phi - 1) + 1}{2a_H (\sigma \phi - 1) + 1} \right) - 1 \right] =$$

$$[2a_H (\sigma \phi - 1) + 1 - \sigma] \beta \sum_{j=0}^{\infty} \beta^j E_t \left[ \left( \hat{Y}_{H,t+j+1} - \hat{Y}_{F,t+j+1} \right) - \left( \hat{Y}_{H,t+j} - \hat{Y}_{F,t+j} \right) \right]$$

Similarly, we can derive the response of NFAs as a function of exogenous shocks.

Under PPI price stability the output gap obeys the following relation,

$$(\hat{Y}_{H,t} - \hat{Y}_{F,t}) = - (1 - a_H) \frac{1 + 2a_H (\sigma \phi - 1)}{\eta [4a_H (1 - a_H) (\sigma \phi - 1) + 1] + \sigma} \hat{W}_t,$$

and capital flows are given by:

$$\hat{B}_t = \hat{B}_{t-1} - \frac{(1 - a_H)}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \frac{1}{\beta} \sum_{j=0}^{\infty} \beta^j \left\{ \left( 2a_H (\sigma \phi - 1) + 1 - \sigma \right) E_t \left( \hat{Y}_{H,t+j+1} - \hat{Y}_{F,t+j+1} \right) - \left( \hat{Y}_{H,t+j} - \hat{Y}_{F,t+j} \right) \right\} +$$

As a result, the wealth gap is given by

$$[2a_H (\sigma \phi - 1) + 1 + 2(1 - a_H) (\sigma \phi - 1) + 1 - \sigma] \beta \sum_{j=0}^{\infty} \beta^j \left\{ \left( 2a_H (\sigma \phi - 1) + 1 - \sigma \right) E_t \left( \hat{Y}_{H,t+j+1} - \hat{Y}_{F,t+j+1} \right) - \left( \hat{Y}_{H,t+j} - \hat{Y}_{F,t+j} \right) \right\} =$$

$$[2a_H (\sigma \phi - 1) + 1 - \sigma] \beta \sum_{j=0}^{\infty} \beta^j \left[ \left( 2a_H (\sigma \phi - 1) + 1 - \sigma \right) E_t \left( \hat{Y}_{H,t+j+1} - \hat{Y}_{F,t+j+1} \right) - \left( \hat{Y}_{H,t+j} - \hat{Y}_{F,t+j} \right) \right] +$$
It is useful to compare the coefficient multiplying the demand gap under PPI price stability and the optimal policy for the case $\eta = 0$ and $\sigma = 1$:

\[
\begin{align*}
PPI \text{ coefficient} & = [2a_H (\phi - 1) + 1] [4a_H (1 - a_H) (\phi - 1) + 1] \\
Optimal \text{ coefficient} & = [2a_H (\phi - 1) + 1] \\
& \quad \left[ \frac{4a_H (1 - a_H) (\phi - 1) + 1}{4a_H (1 - a_H) \phi \frac{4a_H^2 (\phi - 1)^2}{[2a_H (\phi - 1) + 1]^2 \beta \delta_2 (1 - \beta \delta_2)}}, \right.
\end{align*}
\]

where we also used the fact that:

\[
1 - \frac{1}{4} \left( 1 + \beta + \left[ \frac{1}{4a_H (1 - a_H) (\phi - 1) + 1} \right] \frac{(1 - \alpha \beta) (1 - \alpha) \theta}{\alpha} \right)^2 - 1 - \beta^2 \delta_2 \delta_1 \\
+ \frac{1}{4} \left( \eta + \left[ \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \frac{(1 - \alpha \beta) (1 - \alpha) \theta}{\alpha} \right)^2 + \beta = 1 - \beta > 0.
\]

The first term in square bracket is positive for $\phi > 1 - 1/2a_H$, while the second term in the square bracket is always positive for both coefficients but larger under the optimal policy for $\phi \neq 1$. Hence, for given shocks, the demand gap has always the same sign under both policies. Moreover, as its coefficient is larger when positive and smaller when negative, the demand gap is always smaller in absolute value under the optimal policy.

We next proceed to derive expressions for the output gap under PPI price stability and under the optimal policy such that we compare outcomes. Under PPI price stability the output gap is given by,

\[
E_t \left( Y_{H,t+j} - \tilde{Y}_{H,t+j}^b \right) = - (1 - a_H) [1 + 2a_H (\phi - 1)] \tilde{\nu}_t,
\]

In contrast, under the optimal policy, the output gap is given by,

\[
E_t \left( \hat{Y}_{H,t+j} - \tilde{Y}_{H,t+j}^b \right) = - (1 - a_H) [2a_H (\phi - 1) + 1] \\
\left[ \frac{2a_H \phi}{2a_H (\phi - 1) + 1} \frac{\delta_1^i}{\beta \delta_2} + \frac{\beta \delta_2 - 1 - \delta_1^{i+1}}{1 - \delta_1} \right] \tilde{\nu}_t,
\]

under the optimal policy, respectively. Therefore for given shocks, the responses of the output gap are given by

\[
\begin{align*}
\left( \hat{Y}_{H,t} - \tilde{Y}_{H,t}^b \right) & = - (1 - a_H) \quad \text{shocks} \\
\left( \hat{Y}_{H,t} - \tilde{Y}_{H,t}^b \right) & = - (1 - a_H) \quad \text{shocks}
\end{align*}
\]

\[
\begin{align*}
\left[ \frac{2a_H \phi}{2a_H (\phi - 1) + 1} \frac{\delta_1^i}{\beta \delta_2} + \frac{\beta \delta_2 - 1 - \delta_1^{i+1}}{1 - \delta_1} \right] \tilde{\nu}_t.
\end{align*}
\]
On impact, the output gap is smaller in absolute value under the optimal policy. Observe that this is generically true since,
\[
\frac{4a_H (1 - a_H) (\phi - 1) + 1 + 4a_H (1 - a_H) \phi \frac{4a_H^2 (\phi - 1)^2}{2a_H (\phi - 1) + 1} \frac{(1 - \beta)}{\beta \delta_2 (1 - \beta \delta_1)}}{4a_H (1 - a_H) (\phi - 1) + 1} \geq 1 - \frac{1 + 4a_H (1 - a_H) (\phi - 1)}{\beta \delta_2 (1 - \beta \delta_1)}.
\]
is always satisfied. The left-hand side is always larger than 1, while the right hand side is positive but lower than 1, since \(\beta \delta_2 \geq 1\) and
\[
[2a_H (\phi - 1) + 1]^2 \geq 1 + 4a_H (1 - a_H) (\phi - 1).
\]
This establishes that for any value of the elasticity, the optimal policy trades-off more inflation volatility for more stability in the output gap and in the demand gap.

Finally, also for the case \(\eta = 0\) and \(\sigma = 1\) we can derive expressions for capital flows under PPI stability and under the optimal policy,
\[
\hat{B}_t = \hat{B}_{t-1} - \frac{(1 - a_H)}{4a_H (1 - a_H) (\phi - 1) + 1} \beta \sum_{j=0}^{\infty} \beta^j \left\{ \begin{array}{l}
(2a_H (\phi - 1)) E_t \left( (\tilde{Y}_{H,t+j+1}^{fb} - \tilde{Y}_{F,t+j+1}^{fb}) - (\tilde{Y}_{H,t+j}^{fb} - \tilde{Y}_{F,t+j}^{fb}) \right) + \\
-(2a_H (\phi - 1) + 1) E_t \left( (\tilde{\xi}_{C,t+1+j}^{\text{a}} - \tilde{\xi}_{C,t+1+j}^{\text{a}}) - (\tilde{\xi}_{C,t+j}^{\text{a}} - \tilde{\xi}_{C,t+j}^{\text{a}}) \right) \end{array} \right\},
\]
and under the optimal policy:
\[
\hat{B}_t = \hat{B}_{t-1} - \frac{(1 - a_H)}{4a_H (1 - a_H) (\phi - 1) + 1} \beta \sum_{j=0}^{\infty} \beta^j \left\{ \begin{array}{l}
(2a_H (\phi - 1)) E_t \left( (\tilde{Y}_{H,t+j+1}^{fb} - \tilde{Y}_{F,t+j+1}^{fb}) - (\tilde{Y}_{H,t+j}^{fb} - \tilde{Y}_{F,t+j}^{fb}) \right) - \\
(2a_H (\phi - 1) + 1) E_t \left( (\tilde{\xi}_{C,t+1+j}^{\text{a}} - \tilde{\xi}_{C,t+1+j}^{\text{a}}) - (\tilde{\xi}_{C,t+j}^{\text{a}} - \tilde{\xi}_{C,t+j}^{\text{a}}) \right) \end{array} \right\}
\]
\[
+ \frac{4a_H (1 - a_H) (\phi - 1)}{1 - \beta \delta_1} \left( \begin{array}{l}
\hat{W}_t + \\
2 (1 - a_H) \left[ \frac{2a_H (\phi - 1)}{4a_H (1 - a_H) (\phi - 1) + 1} \right] \beta \left( 1 - \delta_1 \right) \hat{Y}_{H,t-1}^{fb} - \tilde{Y}_{H,t-1}^{fb} \end{array} \right).
\]
Given the optimal solution for \(\hat{W}_t\), relative to PPI price stability, expected shocks are now multiplied by the term
\[
- \frac{(1 - a_H)}{4a_H (1 - a_H) (\phi - 1) + 1} \left[ \frac{1 - \frac{1}{(1 - \beta \delta_1) \delta_1}}{(1 - \beta \delta_1) \delta_2} \right].
\]

\[
\frac{4a_H (1 - a_H) (\phi - 1) \left[ 4a_H^2 (\phi - 1) [(\phi - 1)(\beta \delta_2 \delta_1 - 1) - (1 - \delta_1)] + (1 + 4a_H (\phi - 1)) (\beta \delta_2 - 1) \delta_1 \right]}{[2a_H (\phi - 1) + 1]^2 \left[ 4a_H (1 - a_H) (\phi - 1) + 1 + 4a_H (1 - a_H) \phi \frac{4a_H^2 (\phi - 1)^2}{2a_H (\phi - 1) + 1} \frac{(1 - \beta)}{\beta \delta_2 (1 - \beta \delta_1)} \right]}.
\]
since $\beta \delta_2 \delta_1 = 1$ the above further simplifies:

$$
\frac{(1 - a_H)}{4a_H (1 - a_H) (\phi - 1) + 1} \left[ \frac{1 - \delta_1}{\delta_2 - 1} \frac{4a_H (1 - a_H) (\phi - 1)}{2a_H (\phi - 1) + 1} \right]^2.
$$

The second term in brackets is positive for $\phi > 1$ and always less than 1 in absolute value, since $\delta_2 - 1 > 1 - \delta_1$:

$$
\delta_2 - 1 = \frac{1}{2\beta} \left( 1 - \beta + \left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \frac{(1 - \alpha \beta) (1 - \alpha \theta)}{\alpha} \right) +
$$

$$
\frac{1}{2\beta} \sqrt{\left( 1 + \beta + \left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \frac{(1 - \alpha \beta) (1 - \alpha \theta)}{\alpha} \right)^2 - 4\beta},
$$

$$
1 - \delta_1 = \frac{1}{2\beta} \left( \beta - 1 - \left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \frac{(1 - \alpha \beta) (1 - \alpha \theta)}{\alpha} \right) +
$$

$$
\frac{1}{2\beta} \sqrt{\left( 1 + \beta + \left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \frac{(1 - \alpha \beta) (1 - \alpha \theta)}{\alpha} \right)^2 - 4\beta},
$$

which implies that,

$$
1 + \left[ \eta + \frac{\sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \right] \frac{(1 - \alpha \beta) (1 - \alpha \theta)}{\alpha} \geq \delta_2 - 1 = 1 - \delta_1 \Leftrightarrow \frac{(1 - \alpha \beta) (1 - \alpha \theta)}{\alpha} > \beta.
$$

Therefore, optimal policy dampens capital flows for $\phi > 1$ and makes them larger in absolute value for $\phi < 1$.

### 2.2.2  LCP

In the LCP case, the monetary authority minimizes (1), with respect to its arguments $Y_{H,t}$, $Y_{F,t}$, $\Delta_t$, $\bar{W}_t$, and $\pi_{H,t}$, $\pi^*_H$, $\pi^*_F$, subject to the following constraints arising from the competitive equilibrium:

1. NK Phillips curves determining inflation rates

$$
\pi_{H,t} - \beta E_{t} \pi_{H,t+1} =
$$

$$
\frac{(1 - \alpha \beta) (1 - \alpha \theta)}{\alpha} \left[ \frac{(\sigma + \eta) \left( Y_{H,t} - \hat{Y}_{H,t}^b \right) + \bar{\mu}_t +}{ - (1 - a_H) \left[ 2a_H \left( \sigma \phi - 1 \right) \left( \hat{F}_t - \hat{T}_t^b + \Delta_t \right) - \left( \hat{\Delta}_t + \bar{W}_t \right) \right] } \right]
$$

$$
= \pi^*_{H,t} - \beta E_{t} \pi^*_{H,t+1} + \frac{(1 - \alpha \beta) (1 - \alpha \theta)}{\alpha} \Delta_t,
$$
of fundamental shocks are as follows:

\[
\pi_{F,t} - \beta E_t \pi_{F,t+1} = \frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \left[ (\sigma + \eta) \left( \hat{\pi}_{F,t} - \hat{\pi}_{F,t}^b \right) + \hat{\beta}_t^* + \left( \sigma^* \right) \left( \hat{\pi}_{F,t} - \hat{\pi}_{F,t}^b \right) + \left( \sigma^* \right) \left( \hat{\pi}_{F,t} - \hat{\pi}_{F,t}^b \right) \right]
\]

\[
= \pi_{F,t} - \beta E_t \pi_{F,t+1} - \frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \hat{\Delta}_t,
\]

and the constraint on inflation differentials in the same currency:

\[
\pi_{F,t} - \pi_{H,t} - \left( \hat{\pi}_{t} - \hat{\pi}_{t-1} + \hat{\Delta}_t - \hat{\Delta}_t - \hat{\Delta}_t \right) = 0,
\]

where the equilibrium relations for first best outcomes \( \hat{\pi}_{H,t}, \hat{\pi}_{F,t}, \hat{\pi}_{F,t}^b \) in terms of fundamental shocks are as follows:

1. the definition of wealth gap \( \hat{W}_t \) from the difference in budget constraints, \( \hat{\Delta}_t \) and demand gaps:

\[
\hat{\pi}_{F,t} + \hat{\Delta}_t - \hat{\pi}_{F,t}^b = \frac{\left( \hat{Y}_{H,t} - \hat{\pi}_{F,t}^b \right) - \left( \hat{Y}_{H,t} - \hat{\pi}_{F,t}^b \right) - (2aH - 1)(\hat{W}_t + \hat{\Delta}_t)}{4aH (1 - aH) (\sigma \phi - 1) + 1};
\]

2. the equilibrium condition linking relative prices to output gap differentials, \( \hat{\Delta}_t \) and demand gaps:

\[
\hat{\pi}_{F,t} + \hat{\Delta}_t - \hat{\pi}_{F,t}^b = \frac{\left( \hat{Y}_{H,t} - \hat{\pi}_{F,t}^b \right) - \left( \hat{Y}_{H,t} - \hat{\pi}_{F,t}^b \right) - (2aH - 1)(\hat{W}_t + \hat{\Delta}_t)}{4aH (1 - aH) (\sigma \phi - 1) + 1};
\]

3. the definition of wealth gap \( \hat{W}_t \) from the difference in budget constraints, depending also on net wealth \( \hat{B}_t \):

\[
\hat{\pi}_{F,t} + \hat{\Delta}_t - \hat{\pi}_{F,t}^b = \frac{\left( \hat{Y}_{H,t} - \hat{\pi}_{F,t}^b \right) - \left( \hat{Y}_{H,t} - \hat{\pi}_{F,t}^b \right) - (2aH - 1)(\hat{W}_t + \hat{\Delta}_t)}{4aH (1 - aH) (\sigma \phi - 1) + 1};
\]

4. the Euler equation characterizing the evolution of \( \hat{W}_t \) (and thus net wealth \( \hat{B}_t \)):

\[
E_t \hat{W}_t + 1 - \hat{W}_t = 0;
\]
5. finally, in the case of nontrivial portfolio decisions (ω̂i ≠ 0), the (higher-order) Euler equations characterizing these choices and ex-ante excess returns E_t−1 \( \left( \hat{R}_{t,t} - (1+r_t) \right) \) (but this is another matter, see Dedola & Lombardo).

**Bond economy**

We proceed analogously to the previous section, but the LCP economy carries additional complications in the form of law of one price deviations. Observe that in the case of a bond economy, the program amounts to choosing \( b_Y^{H,t}; b_Y^{F,t}; b_t; c_W^t; H_t; F_t; b_B^t \), subject to the following expression for \( \hat{W}_t \) in terms of differences of budget constraints, namely:

\[
(1 - a_H) [2a_H (\phi - 1) + 1] \hat{W}_t = \left[ 4a_H (1 - a_H)(\sigma \phi - 1) + 1 \right] \left( \beta^{-1} \hat{B}_{t-1} - \hat{B}_t \right) + \\
(1 - a_H) [2a_H (\sigma \phi - 1) + 1 - \sigma] \left[ (\hat{Y}_{H,t} - \hat{Y}_{H,t}^b) - (\hat{Y}_{F,t} - \hat{Y}_{F,t}^b) \right] + \\
2a_H (1 - a_H) [2 (1 - a_H)(\sigma \phi - 1) + 1 - \phi] \hat{\Delta}_t + \\
(1 - a_H) [4a_H (1 - a_H)(\sigma \phi - 1) + 1] \cdot \\
\sigma^{-1} \left[ (2a_H (\sigma \phi - 1) + 1 - \sigma) \hat{T}_t^b - (\hat{\zeta}_{C,t} - \hat{\zeta}_{C,t}^b) \right] 
\]

The necessary FOC’s with respect to inflation are given by:

\[
\pi_{H,t} : 0 = -\theta \alpha (1 - a_H) \pi_{H,t} - \gamma_{H,t} + \gamma_{H,t-1} - \gamma_t \\
\pi^*_H : 0 = -\theta \alpha (1 - a_H) \pi^*_H - \gamma^*_H + \gamma^*_H - \gamma_{H,t-1} \\
\pi_{F,t} : 0 = -\theta \alpha (1 - a_H) \pi_{F,t} - \gamma_{F,t} + \gamma_{F,t-1} + \gamma_t \\
\pi^*_F : 0 = -\theta \alpha (1 - a_H) \pi^*_F - \gamma^*_F + \gamma^*_F - \gamma_{F,t-1} 
\]

where \( \gamma_{H,t}; \gamma_{F,t}; \gamma^*_H; \gamma^*_F \) are the multipliers associated with the Phillips curves — whose lags appear reflecting the assumption of commitment, implying the following solutions for the multipliers:

\[
-\frac{(1 - a_H) (1 - a)}{\alpha} \gamma_{H,t} = \theta (a_H \hat{p}_{H,t} + (1 - a_H) \hat{p}_{F,t}) \\
-\frac{(1 - a_H) (1 - a)}{\alpha} \gamma^*_H = \theta (a_H \hat{p}_{F,t} + (1 - a_H) \hat{p}_{H,t}) \\
-2 \frac{(1 - a_H) (1 - a)}{\alpha} \gamma_t = \theta \left[ a_H (\pi_{H,t} - \pi^*_F) + (1 - a_H) (\pi^*_H - \pi_{F,t}) \right] + \\
- \frac{(1 - a_H) (1 - a)}{\alpha} \left[ (-\gamma_{H,t} - \gamma^*_H + \gamma_{H,t-1} + \gamma_{H,t-1}) + \\
- (-\gamma^*_F - \gamma_{F,t} + \gamma_{F,t-1} + \gamma_{F,t-1}) \right].
\]
The FOC with respect to output is given by:

\[ \dot{Y}_{H,t} : 0 = (\sigma + \eta) \left( \dot{Y}_{H,t} - \dot{Y}_{H,t}^{fb} \right) + \]
\[ - \frac{2a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \left[ \left( \dot{Y}_{H,t} - \dot{Y}_{H,t}^{fb} \right) - \left( \dot{Y}_{F,t} - \dot{Y}_{F,t}^{fb} \right) \right] + \]
\[ - \frac{2a_H (1 - a_H) \phi}{2a_H (\phi - 1) + 1} \left( \frac{\Delta_t + \dot{W}_t}{2a_H (\phi - 1) + 1} \right) + \]
\[ - \frac{\sigma + \eta - (1 - a_H) (\sigma - 1)}{2a_H (\phi - 1) + 1} \left( \frac{1 - \alpha \beta}{\alpha} \right) \left( \gamma_{H,t} + \gamma_{H,t}^{*} \right) + \]
\[ - \frac{1}{2a_H (\phi - 1) + 1} \left( \beta E_t \gamma_{t+1} - \gamma_t \right) + \]
\[ - \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \left( \lambda_t - \beta^{-1} \lambda_{t-1} \right); \]

\[ \dot{Y}_{F,t} : 0 = (\sigma + \eta) \left( \dot{Y}_{F,t} - \dot{Y}_{F,t}^{fb} \right) + \]
\[ \frac{2a_H (1 - a_H) (\sigma \phi - 1) \sigma}{4a_H (1 - a_H) (\sigma \phi - 1) + 1} \left[ \left( \dot{Y}_{H,t} - \dot{Y}_{H,t}^{fb} \right) - \left( \dot{Y}_{H,t} - \dot{Y}_{H,t}^{fb} \right) \right] + \]
\[ - \frac{2a_H (1 - a_H) \phi}{2a_H (\phi - 1) + 1} \left( \frac{\Delta_t + \dot{W}_t}{2a_H (\phi - 1) + 1} \right) + \]
\[ - \frac{\sigma + \eta - (1 - a_H) (\sigma - 1)}{2a_H (\phi - 1) + 1} \left( \frac{1 - \alpha \beta}{\alpha} \right) \left( \gamma_{F,t} + \gamma_{F,t}^{*} \right) + \]
\[ - \frac{1}{2a_H (\phi - 1) + 1} \left( \beta E_t \gamma_{t+1} - \gamma_t \right) + \]
\[ - \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \left( \lambda_t - \beta^{-1} \lambda_{t-1} \right); \]

where we have used the fact that

\[ \frac{\partial \dot{W}_t}{\partial Y_{H,t}} = - \frac{\partial \dot{W}_t}{\partial Y_{F,t}} = \frac{2a_H (\sigma \phi - 1) + 1 - \sigma}{2a_H (\phi - 1) + 1} \]
\[ \frac{\partial \dot{F}_t}{\partial Y_{H,t}} = \frac{\sigma - (2a_H - 1) \partial \dot{W}_t}{\partial Y_{H,t}} \]
\[ = - \frac{\partial \dot{F}_t}{\partial Y_{F,t}} = \frac{1}{2a_H (\phi - 1) + 1}; \]

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The FOC with respect to LOOP deviations is given by:

\[
\Delta_t : 0 = -\frac{2a_h (1 - a_h) \phi}{2a_h (\phi - 1) + 1} \left( \Delta_t + \tilde{W}_t \right) + \\
\frac{(1 - \alpha \beta)}{(1 - \alpha)} \frac{1}{4a_h (1 - a_h)(\phi \sigma - 1) + 1} \left( 4a_h (1 - a_h)(\phi \sigma - 1) + 1 \right) \left( \gamma_{H,t} + \gamma_{F,t} - \gamma_{H,t}^* - \gamma_{F,t}^* \right) - \frac{1}{2} \left( \frac{(2a_h - 1) - 2(1 - a_h)[2a_h(\sigma \phi - 1) + 1]}{2a_h(\phi - 1) + 1} \right) \left( \frac{2a_h[2(1 - a_h)(\sigma \phi - 1) + 1]}{2a_h(\phi - 1) + 1} \right). \\
\frac{2a_h - 1}{2a_h(\phi - 1) + 1} \left( \beta E_t \gamma_{t+1} - \gamma_t \right) - \frac{2a_h[2(1 - a_h)(\sigma \phi - 1) + 1]}{2a_h(\phi - 1) + 1} (\lambda_t - \beta^{-1} \lambda_{t-1}),
\]

where we have used the fact that:

\[
\frac{\partial \tilde{W}_t}{\partial \Delta_t} = \frac{2a_h[2(1 - a_h)(\sigma \phi - 1) + 1]}{2a_h(\phi - 1) + 1} \\
= -1 + \frac{4a_h (1 - a_h)(\sigma \phi - 1) + 1}{2a_h(\phi - 1) + 1}
\]

\[
\frac{\partial \tilde{T}_t}{\partial \Delta_t} = -1 - \frac{(2a_h - 1)}{4a_h(1 - a_h)(\sigma \phi - 1) + 1} \cdot \frac{1 + \frac{\partial \tilde{W}_t}{\partial \Delta_t}}{\partial \Delta_t} \\
= -1 - \frac{(2a_h - 1)}{2a_h(\phi - 1) + 1} = -\frac{2a_h \phi}{2a_h(\phi - 1) + 1}
\]

Finally, the FOC with respect to net wealth is given by:

\[
\tilde{B}_t : 0 = 2a_h (1 - a_h) \phi \left[ E_t \tilde{W}_{t+1} - \tilde{W}_t \right] + \\
- (1 - a_h) [2a_h(\sigma \phi - 1) + 1] \frac{(1 - \alpha \beta)}{(1 - \alpha)} \left( \gamma_{H,t} + \gamma_{H,t}^* \right) - \left( \gamma_{H,t} + \gamma_{H,t}^* \right) - \left( \gamma_{F,t} + \gamma_{F,t}^* \right) \\
[(E_t \gamma_{H,t+1} + \gamma_{H,t+1}^*) - (\gamma_{H,t} + \gamma_{H,t}^*)] - \left( E_t \gamma_{F,t+1} + \gamma_{F,t+1}^* \right) - \left( \gamma_{F,t} + \gamma_{F,t}^* \right) \\
(2a_h - 1) \left[ \beta E_t \gamma_{t+2} - E_t \gamma_{t+1} + \beta E_t \gamma_{t+1} - \gamma_t \right] + \\
[4a_h(1 - a_h)(\sigma \phi - 1) + 1] \left[ (E_t \lambda_{t+1} - \beta^{-1} \lambda_t) - (\lambda_t - \beta^{-1} \lambda_{t-1}) \right],
\]

which simplifies as follows:

\[
0 = - (1 - a_h) [2a_h(\sigma \phi - 1) + 1] \frac{(1 - \alpha \beta)}{(1 - \alpha)} \left( \gamma_{H,t} + \gamma_{H,t}^* \right) - \left( \gamma_{H,t} + \gamma_{H,t}^* \right) - \left( \gamma_{F,t} + \gamma_{F,t}^* \right) \\
[(E_t \gamma_{H,t+1} + \gamma_{H,t+1}^*) - (\gamma_{H,t} + \gamma_{H,t}^*)] - \left( E_t \gamma_{F,t+1} + \gamma_{F,t+1}^* \right) - \left( \gamma_{F,t} + \gamma_{F,t}^* \right) \\
(2a_h - 1) \left[ \beta E_t \gamma_{t+2} - E_t \gamma_{t+1} + \beta E_t \gamma_{t+1} - \gamma_t \right] + \\
[4a_h(1 - a_h)(\sigma \phi - 1) + 1] \left[ (E_t \lambda_{t+1} - \beta^{-1} \lambda_t) - (\lambda_t - \beta^{-1} \lambda_{t-1}) \right].
\]
The solution can be expressed in terms of a familiar sum rule for (the change in) world output gaps and CPI inflation rates (where observe that we have switched to the gap notation, e.g. \( \hat{Y}_{H,t} = \hat{Y}_{H,t} - \hat{Y}_{H,t}^{fb} \)):

\[
0 = \hat{Y}_{H,t} + \hat{Y}_{F,t} + \theta (a_{t} \hat{p}_{H,t} + (1 - a_{t}) \hat{p}_{F,t} + a_{t} \hat{\pi}_{F,t} + (1 - a_{t}) \hat{\pi}_{H,t})
\]

\[
= \left[ \hat{Y}_{H,t} - \hat{Y}_{H,t-1} \right] + \left[ \hat{Y}_{F,t} - \hat{Y}_{F,t-1} \right] + \theta \left[ a_{t} \pi_{H,t} + (1 - a_{t}) \pi_{F,t} + a_{t} \pi_{F,t}^{*} + (1 - a_{t}) \pi_{H,t}^{*} \right],
\]

the same as under complete markets, and a cross-country rule.

The latter is difficult to characterize analytically. For the special case of \( \sigma = 1 \) and \( \eta = 0 \), the difference of the FOCs wrt output and LOOP deviations simplifies as follows:

\[
(2a_{t} - 1) \left[ 1 - \frac{4a_{t} (1 - a_{t}) (\phi - 1)}{4a_{t} (1 - a_{t}) (\phi - 1) + 1} \right] \left( \bar{Y}_{H,t} - \bar{Y}_{F,t} \right) +
\]

\[
(2a_{t} - 1) \frac{4a_{t} (1 - a_{t}) \phi}{4a_{t} (1 - a_{t}) (\phi - 1) + 1} \frac{2a_{t} (\phi - 1)}{2a_{t} (\phi - 1) + 1} \left( \bar{\Delta}_{t} + \bar{W}_{t} \right) +
\]

\[
-(1 - \alpha \beta) \left( 1 - \alpha \right) \frac{a_{t} \pi_{H,t} + (1 - a_{t}) \pi_{F,t}}{\alpha} (2a_{t} - 1) \left[ (\gamma_{H,t} + \gamma_{H,t}^{*}) - (\gamma_{F,t} + \gamma_{F,t}^{*}) \right] \]

\[
= \frac{2(2a_{t} - 1)}{2a_{t} (\phi - 1) + 1} \left[ (\beta E_{t} \gamma_{t+1} - \gamma_{t}) - 2a_{t} (\phi - 1) \left( \lambda_{t} - \beta^{-1} \lambda_{t-1} \right) \right];
\]

\[
- \frac{4a_{t} (1 - a_{t}) \phi}{2a_{t} (\phi - 1) + 1} \left( \bar{\Delta}_{t} + \bar{W}_{t} \right) + \frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \cdot
\]

\[
\left[ (\gamma_{H,t} + \gamma_{F,t} - (\gamma_{F,t}^{*} + \gamma_{F,t}^{*})) \right] - (2a_{t} - 1) \left( \gamma_{H,t} + \gamma_{H,t}^{*} - \gamma_{F,t} - \gamma_{F,t}^{*} \right) \]

\[
= \frac{2(2a_{t} - 1)}{2a_{t} (\phi - 1) + 1} \left[ (\beta E_{t} \gamma_{t+1} - \gamma_{t}) - 2a_{t} (\phi - 1) \left( \lambda_{t} - \beta^{-1} \lambda_{t-1} \right) \right].
\]

From the FOCs for inflation, we know that:

\[
\frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \left( \gamma_{H,t} + \gamma_{F,t} \right) = -\theta (a_{t} \hat{p}_{H,t} + (1 - a_{t}) \hat{p}_{F,t})
\]

\[
\frac{(1 - \alpha \beta) (1 - \alpha)}{\alpha} \left( \gamma_{F,t}^{*} + \gamma_{F,t}^{*} \right) = -\theta (a_{t} \hat{\pi}_{F,t} + (1 - a_{t}) \hat{\pi}_{H,t}).
\]

and we are presented with the following cross-country targeting criterion:

\[
0 = \theta \left[ a_{t} \pi_{H,t} + (1 - a_{t}) \pi_{F,t} - (a_{t} \pi_{F,t}^{*} + (1 - a_{t}) \pi_{H,t}^{*}) \right] + \frac{4a_{t} (1 - a_{t}) \phi}{4a_{t} (1 - a_{t}) (\phi - 1) + 1} \left[ (\Delta_{t} + \hat{W}_{t}) - (\bar{\Delta}_{t-1} + \bar{W}_{t-1}) \right] +
\]

\[
\frac{(2a_{t} - 1)}{4a_{t} (1 - a_{t}) (\phi - 1) + 1} \left[ \left( \hat{Y}_{H,t} - \hat{Y}_{H,t}^{fb} \right) - \left( \hat{Y}_{H,t-1} - \hat{Y}_{H,t-1}^{fb} \right) \right] +
\]

\[
\frac{(2a_{t} - 1)}{4a_{t} (1 - a_{t}) (\phi - 1) + 1} \left[ \left( \hat{Y}_{F,t} - \hat{Y}_{F,t}^{fb} \right) - \left( \hat{Y}_{F,t-1} - \hat{Y}_{F,t-1}^{fb} \right) \right] \right],
\]

39
which is the same as the complete markets target for this parameterization.

Notice that, using:

\[ \frac{(2a_H - 1) \left( \Delta_t + \Delta_t \right)}{4a_H (1 - a_H) (\phi - 1) + 1} = \frac{\left( \hat{Y}_{H,t} - \hat{Y}_{F,t} \right) - \left( \hat{Y}_{H,t} - \hat{Y}_{F,t} \right)}{4a_H (1 - a_H) (\phi - 1) + 1} \]

we get that the targeting criterion could be expressed as a combination of the CPI-inflation and consumption differentials:

\[
0 = \theta \left[ (a_H \pi_{H,t} + (1 - a_H) \pi_{F,t}) - (a_H \pi_{H,t}^* + (1 - a_H) \pi_{H,t}^*) \right] + 
\left[ \hat{W}_t - \hat{W}_{t-1} \right] + \left[ (\hat{Q}_t - \hat{Q}_{t}^{fb}) - (\hat{Q}_{t-1} - \hat{Q}_{t-1}^{fb}) \right]
\]

\[
0 = \theta \left[ (a_H \pi_{H,t} + (1 - a_H) \pi_{F,t}) - (a_H \pi_{H,t}^* + (1 - a_H) \pi_{H,t}^*) \right] + 
\left[ (\hat{C}_t - \hat{C}_t^*) - (\hat{C}_t^{fb} - \hat{C}_t^{fb}) \right] - \left[ (\hat{C}_{t-1} - \hat{C}_{t-1}^*) - (\hat{C}_{t-1}^{fb} - \hat{C}_{t-1}^{fb}) \right].
\]

Taking again the difference in CPI inflation using the NKPC:

\[
a_H \pi_{H,t} + (1 - a_H) \pi_{F,t} - (a_H \pi_{F,t} + (1 - a_H) \pi_{F,t}^*) - 
\beta E_t \left( a_H \pi_{H,t+1} + (1 - a_H) \pi_{F,t+1} \right) - \beta E_t \left( a_H \pi_{F,t+1} + (1 - a_H) \pi_{F,t+1}^* \right) = 
\]

\[
= \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha} \left\{ \left[ \left( \hat{Y}_{H,t} - \hat{Y}_{F,t} \right) - \left( \hat{Y}_{F,t} - \hat{Y}_{F,t} \right) \right] - 
\left( 2a_H - 1 \right) \left( \hat{D}_t + \Delta_t \right) + \mu_t - \hat{\mu}_t + 
\right. 
\left. -4a_H (1 - a_H) (\phi - 1) \left( \hat{D}_t - \hat{D}_t \right) + \Delta_t \right\} + 2(1 - a_H) \Delta_t \]

\[
= \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha} \left\{ \hat{Q}_t - \hat{Q}_t^{fb} + (2a_H - 1) \left[ \mu_t - \hat{\mu}_t + \hat{D}_t \right] \right\},
\]

where we have used the following relation:

\[
\hat{Q}_t - \hat{Q}_t^{fb} = (2a_H - 1) \left( \hat{D}_t - \hat{D}_t^{fb} + \Delta_t \right) + \Delta_t
\]

\[
= (2a_H - 1) \left[ \hat{Y}_{F,t} - \hat{Y}_{F,t}^{fb} \right] - \left( 2a_H - 1 \right) \left( \hat{D}_t + \Delta_t \right) + \Delta_t.
\]

In contrast to a complete markets (CM) economy, a policy that sets CPI inflation rates to zero in response to efficient shocks is not optimal.

Finally, notice that we can also write the CPI inflation differential as a function of consumption differentials:

\[
a_H \pi_{H,t} + (1 - a_H) \pi_{F,t} - (a_H \pi_{F,t} + (1 - a_H) \pi_{F,t}^*) - 
\beta E_t \left( a_H \pi_{H,t+1} + (1 - a_H) \pi_{F,t+1} \right) - \beta E_t \left( a_H \pi_{F,t+1} + (1 - a_H) \pi_{F,t+1}^* \right) = 
\]

\[
= \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha} \left\{ \left[ \hat{C}_t - \hat{C}_t^* \right] - \left( \hat{C}_t^{fb} - \hat{C}_t^{fb} \right) \right\} - 2(1 - a_H) \hat{W}_t + (2a_H - 1) \left[ \mu_t - \hat{\mu}_t \right]
\]

\[40\]
Solving explicitly for the allocation under the optimal policy

We next proceed to solve for the optimal allocations. Consider again the difference of the NKPC:

\[ a_H \pi_{H,t} + (1 - a_H) \pi_{F,t} - (a_H \pi^*_{F,t} + (1 - a_H) \pi^*_{H,t}) - \beta E_t (a_H \pi_{H,t+1} + (1 - a_H) \pi_{F,t+1}) - \beta E_t (a_H \pi^*_{F,t+1} + (1 - a_H) \pi^*_{H,t+1}) = \]

\[ = \frac{(1 - \alpha \beta)}{\alpha} \{ \hat{Q}_t - \hat{Q}^f_t + (2a_H - 1) [\hat{\mu}_t - \hat{\mu}^*_t + \hat{W}_t] \}. \]

We next substitute the relative target rule and derive a difference equation in the misalignment and demand gaps:

\[ a_H \pi_{H,t} + (1 - a_H) \pi_{F,t} - (a_H \pi^*_{F,t} + (1 - a_H) \pi^*_{H,t}) - \beta E_t (a_H \pi_{H,t+1} + (1 - a_H) \pi_{F,t+1}) - \beta E_t (a_H \pi^*_{F,t+1} + (1 - a_H) \pi^*_{H,t+1}) = \]

\[ \theta^{-1} \left\{ \beta E_t \left[ (\hat{W}_{t+1} - \hat{W}_t) + (\hat{Q}_{t+1} - \hat{Q}^f_{t+1}) - (\hat{Q}_t - \hat{Q}^f_t) \right] - \right. \]

\[ \left. (\hat{W}_t - \hat{W}_{t-1}) - \left[ (\hat{Q}_t - \hat{Q}^f_t) - (\hat{Q}_{t-1} - \hat{Q}^f_{t-1}) \right] \right\} = \]

\[ = \frac{(1 - \alpha \beta)}{\alpha} \{ \hat{Q}_t - \hat{Q}^f_t + (2a_H - 1) \hat{W}_t \}. \]

The equation admits the following solution as a function of both current and future values of \( \hat{W}_t \):

\[
\left( \hat{Q}_t - \hat{Q}^f_t \right) = \delta_1 \left( \hat{Q}_{t-1} - \hat{Q}^f_{t-1} \right) - \frac{1}{\beta \delta_2} \sum_{j=0}^{\infty} \delta_2^{-j} E_t (\hat{W}_{t+j} - \hat{W}_{t+j-1}) + 
\]

\[- (2a_H - 1) \frac{(1 - \alpha \beta)}{\alpha} \frac{\theta}{\beta \delta_2} \sum_{j=0}^{\infty} \delta_2^{-j} E_t \hat{W}_{t+j}, \]

where \( 0 < \delta_1 < \beta < 1 < \beta^{-1} < \delta_2 \) are the eigenvalues of the difference equation, solving the standard characteristic equation:

\[ \beta \delta^2 - \left[ 1 + \beta + \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha} \theta \right] \delta + 1 = 0, \]

namely

\[ \delta = \frac{1 + \beta + \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha} \theta}{\beta \delta_2 - \frac{\left[ 1 + \beta + \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha} \theta \right]^2 - 4 \beta}{2 \beta}}. \]

As in the PCP case, we can simplify further by using the law of motion for the wealth gap \( \hat{W}_t, E_t \hat{W}_{t+j} = \hat{W}_t \):

\[
\left( \hat{Q}_{t+j} - \hat{Q}^f_{t+j} \right) = \delta_1 \left( \hat{Q}_{t+j-1} - \hat{Q}^f_{t+j-1} \right) - \frac{1}{\beta \delta_2} \left( \hat{W}_{t+j} - \hat{W}_{t+j-1} \right) + 
\]

\[- (2a_H - 1) \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha} \frac{\theta}{\beta (\delta_2 - 1)} \hat{W}_t. \]
The first term \( \tilde{W}_t - \tilde{W}_{t-1} = 0 \) for \( j \geq 1 \), while it is equal to \( \tilde{W}_t \) for \( j = 0 \); instead the last term represents a constant shifter proportional to \( \tilde{W}_t \) for any \( j \geq 0 \). Furthermore, recalling that,

\[
\left[ \tilde{W}_t - \tilde{W}_{t-1} \right] + \left[ \left( \tilde{Q}_t - \tilde{Q}_t^{fb} \right) - \left( \tilde{Q}_{t-1} - \tilde{Q}_{t-1}^{fb} \right) \right] = \\
\left[ \left( \tilde{C}_t - \tilde{C}_t^* \right) - \left( \tilde{C}_t^{fb} - \tilde{C}_t^{*fb} \right) \right] - \left[ \left( \tilde{C}_{t-1} - \tilde{C}_{t-1}^* \right) - \left( \tilde{C}_{t-1}^{fb} - \tilde{C}_{t-1}^{*fb} \right) \right],
\]

we have that inefficient deviations in cross-country consumption differentials (and thus in CPI inflation) are given by:

\[
\left( \frac{\beta_2 - 1}{\beta \delta_2} \right) \left( \tilde{W}_t - \tilde{W}_{t-1} \right) - (2a_H - 1) \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha} \frac{\theta}{\beta (\delta_2 - 1)} \tilde{W}_t \\
- (1 - \delta_1) \left( \tilde{Q}_{t-1} - \tilde{Q}_{t-1}^{fb} \right) = \\
-2\theta (a_H \pi_{H,t} + (1 - a_H) \pi_{F,t}),
\]

which, interestingly, does not depend on the trade elasticity \( \phi \).

Likewise, we can take the difference between the NKPC for \( \pi_{F,t} - \pi_{H,t} \), and use the relation,

\[
\pi_{F,t} - \pi_{H,t} = \left( \tilde{T}_t - \tilde{T}_{t-1} + \tilde{\Delta}_t - \tilde{\Delta}_{t-1} \right)
\]

to get the following difference equation:

\[
\pi_{F,t} - \pi_{H,t} - \beta E_t (\pi_{F,t+1} - \pi_{H,t+1}) = \\
\left( \tilde{T}_t - \tilde{T}_{t-1} + \tilde{\Delta}_t - \tilde{\Delta}_{t-1} \right) - \beta E_t \left( \tilde{T}_{t+1} - \tilde{T}_t + \tilde{\Delta}_{t+1} - \tilde{\Delta}_t \right) = \\
- (1 - \alpha \beta)(1 - \alpha) \frac{\theta}{\alpha} \left[ \left( \tilde{Y}_{H,t} - \tilde{Y}_{H,t}^{fb} \right) - \left( \tilde{Y}_{F,t} - \tilde{Y}_{F,t}^{fb} \right) - \tilde{\Delta}_t + \\
-2(1 - a_H) \left( 2a_H (\phi - 1) \left( \tilde{T}_t - \tilde{T}_t^{fb} + \tilde{\Delta}_t \right) - (\tilde{\Delta}_t + \tilde{\Delta}_t) \right) \right].
\]

Recall the relation between the output gap and relative prices:

\[
\tilde{T}_t - \tilde{T}_t^{fb} + \tilde{\Delta}_t = \left[ \left( \tilde{Y}_{H,t} - \tilde{Y}_{H,t}^{fb} \right) - \left( \tilde{Y}_{F,t} - \tilde{Y}_{F,t}^{fb} \right) \right] - (2a_H - 1) \left( \tilde{W}_t + \tilde{\Delta}_t \right) \\
\frac{4a_H (1 - a_H) (\phi - 1) + 1}{a_H (1 - a_H) (\phi - 1) + 1}
\]

Using this we can simplify the above difference equation as follows:

\[
\beta E_t \left[ \left( \tilde{T}_{t+1} - \tilde{T}_{t+1}^{fb} + \tilde{\Delta}_{t+1} \right) - \left( \tilde{T}_t - \tilde{T}_t^{fb} + \tilde{\Delta}_t \right) \right] - \\
\left[ \left( \tilde{T}_t - \tilde{T}_t^{fb} + \tilde{\Delta}_t \right) - \left( \tilde{T}_{t-1} - \tilde{T}_{t-1}^{fb} + \tilde{\Delta}_{t-1} \right) \right] - \\
\frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha} \left( \tilde{T}_t - \tilde{T}_t^{fb} + \tilde{\Delta}_t \right) = \\
(1 - \alpha \beta)(1 - \alpha) \frac{\theta}{\alpha} \tilde{W}_t - E_t \left[ \beta \left( \tilde{T}_{t+1}^{fb} - \tilde{T}_t^{fb} \right) - \left( \tilde{T}_t^{fb} - \tilde{T}_{t-1}^{fb} \right) \right].
\]
We solve this difference equation for \((\bar{T}_t - \bar{T}_t^{fb} + \bar{\Delta}_t)\):

\[
(\bar{T}_t - \bar{T}_t^{fb} + \bar{\Delta}_t) = \nu_1 \left(\bar{T}_{t-1} - \bar{T}_{t-1}^{fb} + \bar{\Delta}_{t-1}\right) - \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha \beta} \sum_{j=0}^{\infty} \nu_2^{-j-1} \bar{W}_t + \sum_{j=0}^{\infty} \nu_2^{-j-1} E_t \left[\left(\bar{T}_{t+j+1}^{fb} - \bar{T}_{t+j}^{fb}\right) - \beta^{-1} \left(\bar{T}_{t+j+1}^{fb} - \bar{T}_{t+j+1}^{fb}\right)\right],
\]

where \(0 < \nu_1 < 1 < \beta^{-1} < \nu_2\) are the eigenvalues of the difference equation, solving the standard characteristic equation:

\[
\beta \nu^2 - \left[1 + \beta + \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha}\right] \nu + 1 = 0,
\]

namely:

\[
\nu = \frac{1 + \beta + \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha} \pm \sqrt{\left[1 + \beta + \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha}\right]^2 - 4 \beta}}{2 \beta}.
\]

We simplify further using the fact that \(\bar{W}_t\) is a martingale:

\[
(\bar{T}_{t+j} - \bar{T}_{t+j}^{fb} + \bar{\Delta}_{t+j}) = \nu_1 \left(\bar{T}_{t+j-1} - \bar{T}_{t+j-1}^{fb} + \bar{\Delta}_{t+j-1}\right) - \frac{(\beta \nu_2 - 1)}{\beta \nu_2} \bar{W}_t + \sum_{s=0}^{\infty} \nu_2^{-s-1} E_{t+j} \left[\left(\bar{T}_{t+j+s+1}^{fb} - \bar{T}_{t+j+s}^{fb}\right) - \beta^{-1} \left(\bar{T}_{t+j+s+1}^{fb} - \bar{T}_{t+j+s+1}^{fb}\right)\right],
\]

where we have also used the fact that: \(\frac{(\beta \nu_2 - 1)}{\beta \nu_2} = \frac{(1 - \alpha \beta)(1 - \alpha)}{\alpha \beta (\nu_2 - 1)}\).

Thus, we also reach a solution for the deviations from the law of one price:

\[
\bar{\Delta}_t = \left(\bar{Q}_t - \bar{Q}_t^{fb}\right) - (2a_H - 1) \left(\bar{T}_t - \bar{T}_t^{fb} + \bar{\Delta}_t\right),
\]

and for the output gap

\[
(\bar{Y}_{H,t} - \bar{Y}_{H,t}^{fb}) - (\bar{Y}_{F,t} - \bar{Y}_{F,t}^{fb}) = 2 \left(\bar{Y}_{H,t} - \bar{Y}_{H,t}^{fb}\right) = 4a_H (1 - a_H) \left(\bar{T}_t - \bar{T}_t^{fb} + \bar{\Delta}_t\right) + (2a_H - 1) \left(\bar{W}_t + \bar{\Delta}_t\right) = 4a_H (1 - a_H) \phi \left(\bar{T}_t - \bar{T}_t^{fb} + \bar{\Delta}_t\right) + (2a_H - 1) \left(\bar{W}_t + \bar{Q}_t - \bar{Q}_t^{fb}\right),
\]

namely:

\[
2 \left(\bar{Y}_{H,t} - \bar{Y}_{H,t}^{fb}\right) = (2a_H - 1) \left(\bar{W}_t + \left(\bar{Q}_t - \bar{Q}_t^{fb}\right)\right) - 4a_H (1 - a_H) \phi \frac{(\beta \nu_2 - 1)}{\beta \nu_2} \bar{W}_t.
\]

\[
4a_H (1 - a_H) \phi \left[\sum_{j=0}^{\infty} \nu_2^{-j-1} E_t \left[\left(\bar{T}_{t+j+1}^{fb} - \bar{T}_{t+j}^{fb}\right) - \beta^{-1} \left(\bar{T}_{t+j+1}^{fb} - \bar{T}_{t+j+1}^{fb}\right)\right] + \nu_1 \left(\bar{T}_{t-1} - \bar{T}_{t-1}^{fb} + \bar{\Delta}_{t-1}\right)\right]
\]
Finally, we can solve for net foreign assets and (the permanent shift in) $\bar{W}_t$ by using the budget constraint:

$$\bar{W}_t = \left[ \begin{array}{c}
2\beta^{-1} \left( \bar{Y}_{H,t} - \bar{Y}_{H,t}^f \right) - \left( \bar{Y}_{F,t} - \bar{Y}_{F,t}^f \right) + \\
-2 (1 - a_H) \left( \bar{T}_t - \bar{T}_t^f \right) - \left( \bar{Q}_t - \bar{Q}_t^f \right)
\end{array} \right] + \\
2 (1 - a_H) \left[ 2a_H (\phi - 1) \bar{T}_t^f - \left( \bar{\zeta}_{C,t} - \bar{\zeta}_{C,t}^* \right) \right].$$

We can also solve for the link between the output gap and relative prices:

$$\left( \bar{Y}_{H,t} - \bar{Y}_{H,t}^f \right) - \left( \bar{Y}_{F,t} - \bar{Y}_{F,t}^f \right) = \\
4a_H (1 - a_H) \phi \left( \bar{T}_t - \bar{T}_t^f + \bar{\Delta}_t \right) + (2a_H - 1) \left( \bar{W}_t + \left( \bar{Q}_t - \bar{Q}_t^f \right) \right),$$

and the relation among the latter:

$$2a_H \left( \bar{T}_t - \bar{T}_t^f + \bar{\Delta}_t \right) = \left( \bar{Q}_t - \bar{Q}_t^f \right) + \left( \bar{T}_t - \bar{T}_t^f \right),$$

We obtain the following simplification:

$$(1 - a_H) \bar{W}_t = \beta^{-1} \left( \bar{W}_{t-1} - \beta \bar{W}_t \right) + \\
2a_H (1 - a_H) (\phi - 1) \left( \bar{T}_t - \bar{T}_t^f + \bar{\Delta}_t \right) + \\
(1 - a_H) \left[ 2a_H (\phi - 1) \bar{T}_t^f - \left( \bar{\zeta}_{C,t} - \bar{\zeta}_{C,t}^* \right) \right].$$

Using the consumption Euler equation, we get the following difference equation with eigenvalues $\beta^{-1}$ and 1, that we can solve explicitly for NFAs:

$$\beta^{-1} \left[ E_t \left( \beta \bar{W}_{t+1} - \bar{W}_t \right) - \left( \beta \bar{W}_t - \bar{W}_{t-1} \right) \right] = \\
2a_H (1 - a_H) (\phi - 1) E_t \left( \left( \bar{T}_{t+1} - \bar{T}_{t+1}^f + \bar{\Delta}_{t+1} \right) - \left( \bar{T}_t - \bar{T}_t^f + \bar{\Delta}_t \right) \right) + \\
(1 - a_H) \left[ 2a_H (\phi - 1) E_t \left( \bar{T}_{t+1}^f - \bar{T}_t^f \right) - E_t \left( \left( \bar{\zeta}_{C,t+1} - \bar{\zeta}_{C,t+1}^* \right) - \left( \bar{\zeta}_{C,t} - \bar{\zeta}_{C,t}^* \right) \right) \right].$$

Namely:

$$\bar{W}_t - \bar{W}_{t-1} = -2a_H (1 - a_H) (\phi - 1) \cdot \\
\left[ \beta \sum_{j=0}^{\infty} \beta^j E_t \left[ \left( \bar{T}_{t+j+1} - \bar{T}_{t+j+1}^f + \bar{\Delta}_{t+j+1} \right) - \left( \bar{T}_{t+j} - \bar{T}_{t+j}^f + \bar{\Delta}_{t+j} \right) \right] \right] - \\
(1 - a_H) \left[ 2a_H (\phi - 1) \beta \sum_{j=0}^{\infty} \beta^j E_t \left[ \bar{T}_{t+j+1}^f - \bar{T}_{t+j}^f \right] \right] + \\
(1 - a_H) \beta \sum_{j=0}^{\infty} \beta^j E_t \left[ \left( \bar{\zeta}_{C,t+j+1} - \bar{\zeta}_{C,t+j+1}^* \right) - \left( \bar{\zeta}_{C,t+j} - \bar{\zeta}_{C,t+j}^* \right) \right].$$
We can further simplify the latter expression using the above solutions for relative price misalignments; namely we have that for $j \geq 0$:

$$E_t \left( \hat{T}_{t+j} - \hat{T}_{t+j}^b + \hat{\Delta}_{t+j} \right) = \nu_{1}^{j+1} \left( \hat{T}_{t-1} - \hat{T}_{t-1}^b + \hat{\Delta}_{t-1} \right) - \frac{1 - \nu_{1}^{j+1} (\beta \nu_{2} - 1)}{1 - \nu_{1}} \nu_{1}^{j} W_t$$

$$+ \sum_{s=0}^{j} \nu_{1}^{j-s} \left( \sum_{h=0}^{\infty} \nu_{2}^{-h-1} E_t \left[ \left( \hat{T}_{t+h+s+1}^b - \hat{T}_{t+h+s}^b \right) - \beta^{-1} \left( \hat{T}_{t+h+s} - \hat{T}_{t+h+s-1}^b \right) \right] \right),$$

Putting the above together we can find the following solution for NFAs:

$$\hat{W}_t - \hat{W}_{t-1} = -2a_H (1 - a_H) (\phi - 1) \cdot$$

$$\left[ \beta \sum_{j=0}^{\infty} \beta^{j} E_t \left( \left( \hat{T}_{t+j+1} - \hat{T}_{t+j+1}^b + \hat{\Delta}_{t+j+1} \right) - \left( \hat{T}_{t+j} - \hat{T}_{t+j}^b + \hat{\Delta}_{t+j} \right) \right) \right] -$$

$$(1 - a_H) [2a_H (\phi - 1)] \beta \sum_{j=0}^{\infty} \beta^{j} E_t \left[ \left( \hat{T}_{t+j+1} - \hat{T}_{t+j}^b \right) \right] +$$

$$(1 - a_H) \beta \sum_{j=0}^{\infty} \beta^{j} E_t \left[ \left( \hat{\zeta}_{C,t+j+1} - \hat{\zeta}_{C,t+j+1}^* \right) - \left( \hat{\zeta}_{C,t+j} - \hat{\zeta}_{C,t+j}^* \right) \right].$$

Simplifying,

$$\hat{W}_t - \hat{W}_{t-1} = 2a_H (1 - a_H) (\phi - 1) (1 - \nu_{1}) \cdot$$

$$\beta \sum_{j=0}^{\infty} \beta^{j} \nu_{1}^{j+1} \left( \hat{T}_{t-1} - \hat{T}_{t-1}^b + \hat{\Delta}_{t-1} \right) +$$

$$\beta \sum_{j=0}^{\infty} \beta^{j} \left\{ \sum_{s=0}^{j} \nu_{1}^{j-s} \left( \sum_{h=0}^{\infty} \nu_{2}^{-h-1} E_t \left[ \left( \hat{T}_{t+h+s+1}^b - \hat{T}_{t+h+s}^b \right) - \beta^{-1} \left( \hat{T}_{t+h+s} - \hat{T}_{t+h+s-1}^b \right) \right] \right) \right\}$$

$$+ 2a_H (1 - a_H) (\phi - 1) \left( \frac{(\beta \nu_{2} - 1)}{\nu_{2} (1 - \beta \nu_{1})} \nu_{1} \hat{D}_t \right)$$

$$- 2a_H (1 - a_H) (\phi - 1) \cdot$$

$$\beta \sum_{j=0}^{\infty} \beta^{j} \left\{ \sum_{s=0}^{\infty} \nu_{2}^{-s-1} E_t \left[ \left( \hat{T}_{t+j+s+2}^b - \hat{T}_{t+j+s+1}^b \right) - \beta^{-1} \left( \hat{T}_{t+j+s+1} - \hat{T}_{t+j+s}^b \right) \right] \right\} -$$

$$(1 - a_H) [2a_H (\phi - 1)] \beta \sum_{j=0}^{\infty} \beta^{j} E_t \left[ \left( \hat{T}_{t+j+1} - \hat{T}_{t+j}^b \right) \right] +$$

$$(1 - a_H) \beta \sum_{j=0}^{\infty} \beta^{j} E_t \left[ \left( \hat{\zeta}_{C,t+j+1} - \hat{\zeta}_{C,t+j+1}^* \right) - \left( \hat{\zeta}_{C,t+j} - \hat{\zeta}_{C,t+j}^* \right) \right].$$

Finally, we can solve for the permanent response of $\hat{W}_t$ as a function only of
exogenous shocks:

\[
(1 - a_H) \tilde{W}_t = \beta^{-1} \left( \tilde{W}_{t-1} - \beta \tilde{W}_t \right) +
2a_H \left( 1 - a_H \right) (\phi - 1) \left( \tilde{T}_t - \tilde{T}_t^{fb} + \tilde{\Delta}_t \right) +
(1 - a_H) \left[ 2a_H (\phi - 1) \tilde{T}_t^{fb} - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^{*} \right) \right].
\]

\[
\left( \tilde{T}_t - \tilde{T}_t^{fb} + \tilde{\Delta}_t \right) = \nu_1 \left( \tilde{T}_{t-1} - \tilde{T}_{t-1}^{fb} + \tilde{\Delta}_{t-1} \right) - \frac{(\beta \nu_2 - 1)}{\beta \nu_2} \tilde{W}_t +
\sum_{s=0}^{\infty} \nu_2^{s-1} E_t \left[ \left( \tilde{T}_{t+s+1}^{fb} - \tilde{T}_{t+s+1}^{fb} \right) - \beta^{-1} \left( \tilde{T}_{t+s}^{fb} - \tilde{T}_{t+s-1}^{fb} \right) \right],
\]

\[
(1 - a_H) \left[ 1 + 2a_H (\phi - 1) \frac{(\beta \nu_2 - 1)}{\beta \nu_2} \right] \tilde{W}_t =
\left( \beta^{-1} \tilde{W}_{t-1} - \tilde{W}_t \right) +
2a_H (1 - a_H)(\phi - 1) \sum_{s=0}^{\infty} \nu_2^{s-1} E_t \left[ \left( \tilde{T}_{t+s+1}^{fb} - \tilde{T}_{t+s+1}^{fb} \right) - \beta^{-1} \left( \tilde{T}_{t+s}^{fb} - \tilde{T}_{t+s-1}^{fb} \right) \right] -
(1 - a_H) \left[ 2a_H (\phi - 1) \tilde{T}_t^{fb} - \left( \tilde{\zeta}_{C,t} - \tilde{\zeta}_{C,t}^{*} \right) \right].
\]

\[
\tilde{W}_1 - \tilde{W}_{t-1} =
2a_H (1 - a_H)(\phi - 1) (1 - \nu_1) \left\{ \beta \sum_{j=0}^{\infty} \beta^j \nu_1^{j+1} \left( \tilde{T}_{t-1} - \tilde{T}_{t-1}^{fb} + \tilde{\Delta}_{t-1} \right) \right\} +
\left[ \beta \sum_{j=0}^{\infty} \beta^j \sum_{s=0}^{j-1} \nu_2^{s-1} \sum_{h=0}^{j-s} \nu_2^{h-1} E_t \left[ \frac{\left( \tilde{T}_{t+h+s+1}^{fb} - \tilde{T}_{t+h+s}^{fb} \right)}{\beta^{-1} \left( \tilde{T}_{t+h+s}^{fb} - \tilde{T}_{t+h+s-1}^{fb} \right)} \right] \right\} -
2a_H (1 - a_H)(\phi - 1) \frac{(\beta \nu_2 - 1)}{\nu_2} \beta \nu_1 \tilde{W}_t -
2a_H (1 - a_H)(\phi - 1) \beta \sum_{j=0}^{\infty} \beta^j \sum_{s=0}^{j-1} \nu_2^{s-1} E_t \left[ \frac{\left( \tilde{T}_{t+j+s+2}^{fb} - \tilde{T}_{t+j+s+1}^{fb} \right)}{\beta^{-1} \left( \tilde{T}_{t+j+s+1}^{fb} - \tilde{T}_{t+j+s}^{fb} \right)} \right] -
(1 - a_H) \left[ 2a_H (\phi - 1) \right] \beta \sum_{j=0}^{\infty} \beta^j E_t \left[ \left( \tilde{T}_{t+j+1} - \tilde{T}_{t+j}^{fb} \right) \right] +
(1 - a_H) \beta \sum_{j=0}^{\infty} \beta^j E_t \left[ \left( \tilde{\zeta}_{C,t+j+1} - \tilde{\zeta}_{C,t+j+1}^{*} \right) - \left( \tilde{\zeta}_{C,t+j} - \tilde{\zeta}_{C,t+j}^{*} \right) \right].
\]

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\[(1 - a_h) \left[ 1 + 2a_H (\phi - 1) \frac{(\beta \nu_2 - 1)}{\beta \nu_2 (1 - \beta \nu_1)} \right] \tilde{W}_t = \]

\[
\beta^{-1} \tilde{W}_{t-1} - \tilde{W}_{t-1} + 2a_H (1 - a_H) (\phi - 1) \beta \sum_{j=0}^{\infty} \beta^j.
\]

\[
\sum_{s=0}^{\infty} \nu_2^{-s-1} E_t \left[ (\tilde{T}_{t+j+s+2}^{fb} - \tilde{T}_{t+j+s+1}^{fb}) - \beta^{-1} (\tilde{T}_{t+j+s+1}^{fb} - \tilde{T}_{t+j+s}^{fb}) \right] - \beta^{-1} (\tilde{T}_{t+j+s+1}^{fb} - \tilde{T}_{t+j+s}^{fb})
\]

\[
(1 - \nu_1) \sum_{s=0}^{\infty} \nu_1^{-j-s} \sum_{h=0}^{\infty} \nu_2^{-h-1} E_t \left[ (\tilde{T}_{t+h+s+1}^{fb} - \tilde{T}_{t+h+s-1}^{fb}) - \beta^{-1} (\tilde{T}_{t+h+s}^{fb} - \tilde{T}_{t+h+s-1}^{fb}) \right]
\]

\[
+ (1 - a_H) \beta \sum_{j=0}^{\infty} \beta^j \left\{ -E_t \left[ (\tilde{\zeta}_{C,t+j+1}^{c} - \tilde{\zeta}_{C,t+j}^{c}) - (\tilde{\zeta}_{C,t+j}^{c} - \tilde{\zeta}_{C,t+j}^{c}) \right] + 2a_H (\phi - 1) E_t \left[ (\tilde{T}_{t+j+1}^{fb} - \tilde{T}_{t+j}^{fb}) \right] \right\}
\]

\[
+ 2a_H (1 - a_H) (\phi - 1) \sum_{s=0}^{\infty} \nu_2^{-s-1} E_t \left[ (\tilde{T}_{t+s+1}^{fb} - \tilde{T}_{t+s}^{fb}) - \beta^{-1} (\tilde{T}_{t+s}^{fb} - \tilde{T}_{t+s-1}^{fb}) \right] + (1 - a_H) \left[ 2a_H (\phi - 1) \tilde{T}_{t}^{fb} - (\tilde{\zeta}_{C,t}^{c} - \tilde{\zeta}_{C,t}^{c}) \right].
\]

\[
2a_H (1 - a_H) (\phi - 1) \frac{(\beta \nu_2 - 1)}{\nu_2 (1 - \beta \nu_1)} \tilde{W}_t = \]

\[
\frac{2a_H (\phi - 1) \frac{(\beta \nu_2 - 1)}{\nu_2 (1 - \beta \nu_1)}}{1 + 2a_H (\phi - 1) \frac{(\beta \nu_2 - 1)}{\nu_2 (1 - \beta \nu_1)}} \left\{ \beta^{-1} \tilde{W}_{t-1} - \tilde{W}_{t-1} + 2a_H (1 - a_H) (\phi - 1) \beta \cdot \right\}
\]

\[
\sum_{j=0}^{\infty} \beta^j \left[ \sum_{s=0}^{\infty} \nu_2^{-s-1} E_t \left[ (\tilde{T}_{t+j+s+2}^{fb} - \tilde{T}_{t+j+s+1}^{fb}) - \beta^{-1} (\tilde{T}_{t+j+s+1}^{fb} - \tilde{T}_{t+j+s}^{fb}) \right] - \beta^{-1} (\tilde{T}_{t+j+s+1}^{fb} - \tilde{T}_{t+j+s}^{fb}) \right] + \right.
\]

\[
(1 - \nu_1) \sum_{s=0}^{\infty} \nu_1^{-j-s} \sum_{h=0}^{\infty} \nu_2^{-h-1} E_t \left[ (\tilde{T}_{t+h+s+1}^{fb} - \tilde{T}_{t+h+s-1}^{fb}) - \beta^{-1} (\tilde{T}_{t+h+s}^{fb} - \tilde{T}_{t+h+s-1}^{fb}) \right] \]

\[
2a_H (1 - a_H) (\phi - 1) \sum_{s=0}^{\infty} \nu_2^{-s-1} E_t \left[ (\tilde{T}_{t+s+1}^{fb} - \tilde{T}_{t+s}^{fb}) - \beta^{-1} (\tilde{T}_{t+s}^{fb} - \tilde{T}_{t+s-1}^{fb}) \right] + (1 - a_H) \left[ 2a_H (\phi - 1) \tilde{T}_{t}^{fb} - (\tilde{\zeta}_{C,t}^{c} - \tilde{\zeta}_{C,t}^{c}) \right].
\]
\[
\begin{aligned}
\tilde{W}_t - \tilde{W}_{t-1} &= \\
2a_H (1 - a_H) (\phi - 1) (1 - \nu_1) \beta \sum_{j=0}^{\infty} \beta^j \nu_1^{j+1} \left( \tilde{T}_{t-1} - \tilde{T}^b_{t-1} + \Delta_{t-1} \right) + \\
\frac{2a_H (\phi - 1) \left( \frac{\beta \nu_2 - 1}{\nu_2 (1 - 3 \nu_1)} \right)}{1 + 2a_H (\phi - 1) \left( \frac{\beta \nu_2 - 1}{\nu_2 (1 - 3 \nu_1)} \right)} \left( \beta^{-1} \tilde{W}_{t-1} - \tilde{W}_{t-1} \right) - \\
\frac{2a_H (\phi - 1) \left( \frac{\beta \nu_2 - 1}{\nu_2} \beta \nu_1 \nu_2 \right)}{1 + 2a_H (\phi - 1) \left( \frac{\beta \nu_2 - 1}{\nu_2 (1 - 3 \nu_1)} \right)} \left[ 2a_H (1 - a_H) (\phi - 1) \right].
\end{aligned}
\]

\[
\begin{aligned}
\sum_{j=0}^{\infty} \beta^j \left\{ (1 - \nu_1) \beta \sum_{s=0}^{\nu_1^j-1} \sum_{s=0}^{\nu_1^{j-s}} \sum_{s=0}^{\nu_1^{j-s}} E_t \left[ \frac{\tilde{T}^b_{t+j+s+1} - \tilde{T}^b_{t+j+s}}{\beta^{-1} \left( \tilde{T}^b_{t+j+s} - \tilde{T}^b_{t+j+s-1} \right)} \right] \right\} - \\
\frac{1 + 2a_H (\phi - 1) \left( \frac{\beta \nu_2 - 1}{\nu_2} \beta \nu_1 \nu_2 \right)}{1 + 2a_H (\phi - 1) \left( \frac{\beta \nu_2 - 1}{\nu_2 (1 - 3 \nu_1)} \right)} (1 - a_H) \beta .
\end{aligned}
\]

\[
\begin{aligned}
\sum_{j=0}^{\infty} \beta^j \left\{ 2a_H (\phi - 1) E_t \left[ \frac{\tilde{\zeta}_{C,t+j+1} - \tilde{\zeta}^s_{C,t+j+1}}{\left( \tilde{\zeta}_{C,t+j+1} - \tilde{\zeta}^s_{C,t+j+1} \right)} \right] + \\
2a_H (1 - a_H) (\phi - 1) \sum_{s=0}^{\nu_1^{j-s-1}} E_t \left[ \frac{\tilde{T}^b_{t+s+1} - \tilde{T}^b_{t+s}}{\beta^{-1} \left( \tilde{T}^b_{t+s} - \tilde{T}^b_{t+s-1} \right)} \right] + \\
\frac{2a_H (\phi - 1) \left( \frac{\beta \nu_2 - 1}{\nu_2 (1 - 3 \nu_1)} \right)}{1 + 2a_H (\phi - 1) \left( \frac{\beta \nu_2 - 1}{\nu_2 (1 - 3 \nu_1)} \right)} (1 - a_H) \left[ 2a_H (\phi - 1) \tilde{T}^b_t - \tilde{\zeta}_{C,t} - \tilde{\zeta}^s_{C,t} \right].
\end{aligned}
\]
Furthermore,

\[ (1 - a_H) \left[ 1 + 2a_H (\phi - 1) \frac{(\beta \nu_2 - 1)}{\beta \nu_2 (1 - \beta \nu_1)} \right] \hat{W}_t = \]

\[ \beta^{-1} \hat{W}_{t-1} - \hat{W}_{t-1} + 2a_H (1 - a_H) (\phi - 1) \sum_{j=0}^{\infty} \beta^j . \]

\[
\left\{ \begin{array}{l}
\sum_{s=0}^{\infty} \nu_2^{s-1} E_t \left[ \frac{\left( \bar{T}_{t+j+s+1}^{fb} - \bar{T}_{t+j+s}^{fb} \right)^2}{\beta^{-1} \left( \bar{T}_{t+j+s}^{fb} - \bar{T}_{t+j+s-1}^{fb} \right)} \right] \\
(1 - \nu_1) \beta \sum_{s=0}^{j} \nu_1^{j-s} \sum_{h=0}^{\infty} \nu_2^{h-1} E_t \left[ \frac{\left( \bar{T}_{t+h+s+1}^{fb} - \bar{T}_{t+h+s}^{fb} \right)^2}{\beta^{-1} \left( \bar{T}_{t+h+s}^{fb} - \bar{T}_{t+h+s-1}^{fb} \right)} \right]
\end{array} \right\}
\]

\[ (1 - a_H) \beta \sum_{j=0}^{\infty} \beta^j \left\{ \begin{array}{l}
2a_H (\phi - 1) E_t \left[ \left( \bar{T}_{t+j+1}^{fb} - \bar{T}_{t+j}^{fb} \right) \right] - \\
E_t \left[ \left( \hat{\zeta}_{C,t+j+1} - \hat{\zeta}_{C,t+j} \right) - \left( \hat{\zeta}_{C,t} \right) \right]
\end{array} \right\} + \\
(1 - a_H) \left[ 2a_H (\phi - 1) \bar{T}_{t}^{fb} - \left( \hat{\zeta}_{C,t} \right) \right].
\]

Lastly, we derive the link between the demand gap and capital flows:

\[ (1 - a_H) \left[ 1 + 2a_H (\phi - 1) \frac{(\beta \nu_2 - 1)}{\beta \nu_2} \right] \hat{W}_t = -\hat{W}_t + \\
(1 - a_H) 2a_H (\phi - 1) \sum_{s=0}^{\infty} \nu_2^{s-1} E_t \left[ \left( \bar{T}_{t+s+1}^{fb} - \bar{T}_{t+s}^{fb} \right) - \beta^{-1} \left( \bar{T}_{t+s}^{fb} - \bar{T}_{t+s-1}^{fb} \right) \right] + \\
\left[ \frac{1 + 2a_H (\phi - 1) \frac{(\beta \nu_2 - 1) \nu_1}{\nu_2 (1 - \beta \nu_1)}}{1 + 2a_H (\phi - 1) \frac{(\beta \nu_2 - 1) \nu_1}{\beta \nu_2 (1 - \beta \nu_1)}} \right] (1 - a_H) \left[ 2a_H (\phi - 1) \bar{T}_{t}^{fb} - \left( \hat{\zeta}_{C,t} \right) \right].\]